EE-2025 Spring-00

Lecture 24

H(z) & Frequency Response

21-April-00

Info: Web-CT, Lab, HW

- **Final Exam, 2–May @ 11:30 AM**
  - Calculator, 1 page Handwritten Notes
  - **Review: Monday, 1–May @ 6:45 pm**
- **HW #13 due last day (in Lecture)**
- **Labs DEADLINE**
  - **ALL Lab Reports due by 28–April, no later than Friday at 5 pm.**

READING ASSIGNMENTS

- **This Lecture:**
  - Chapter 8, pp. 263–279
- **Other Reading:**
  - Recitation: Ch. 8, pp. 261–272
    - POLES & ZEROS
  - Next Lecture: Chapter 8, pp. 279–300

LECTURE OBJECTIVES

- **SYSTEM FUNCTION: H(z)**
- **H(z) has POLES and ZEROS**
- **FREQUENCY RESPONSE of IIR**
  - Get H(z) first

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

- **THREE–DOMAIN APPROACH**

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

ZZZZZ-Transform

IIR FILTER REVIEW

- **ADD PREVIOUS OUTPUTS**

$$y[n] = 0.8y[n-1] + 3x[n]$$

FEEDBACK COEFFICIENT

SIGN CHANGE

- **MATLAB**
  - `yy = filter([3],[1,-0.8],xx)`

$$H(z) = \frac{3}{1-0.8z^{-1}}$$

IMPULSE RESPONSE

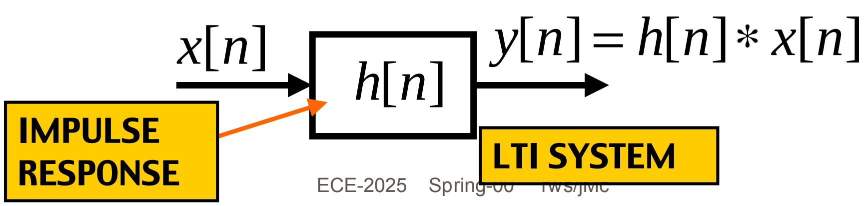
- **DIFFERENCE EQUATION:**

$$y[n] = 0.8y[n-1] + 3x[n]$$

- **Find h[n]**

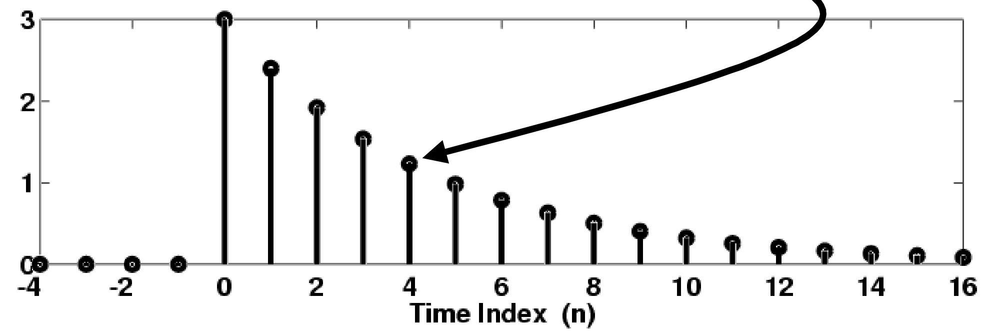
$$h[n] = 3(0.8)^n u[n]$$

- **CONVOLUTION in TIME-DOMAIN**



PLOT IMPULSE RESPONSE

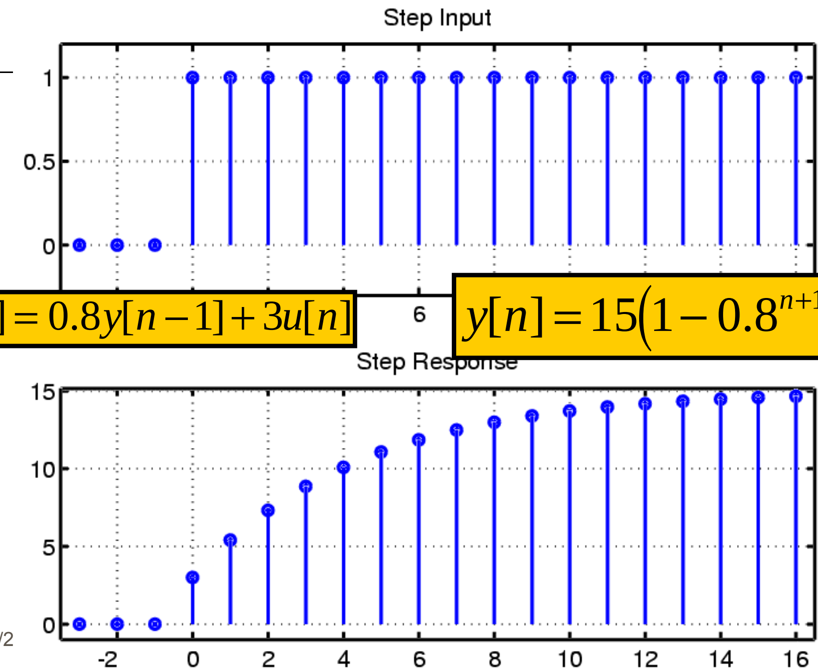
$$h[n] = b_0 (a_1)^n u[n] = 3(0.8)^n u[n]$$



PLOT STEP RESPONSE

$$y[n] = 0.8y[n-1] + 3u[n]$$

$$y[n] = 15\left(1 - 0.8^{n+1}\right)u[n]$$



THREE DOMAINS

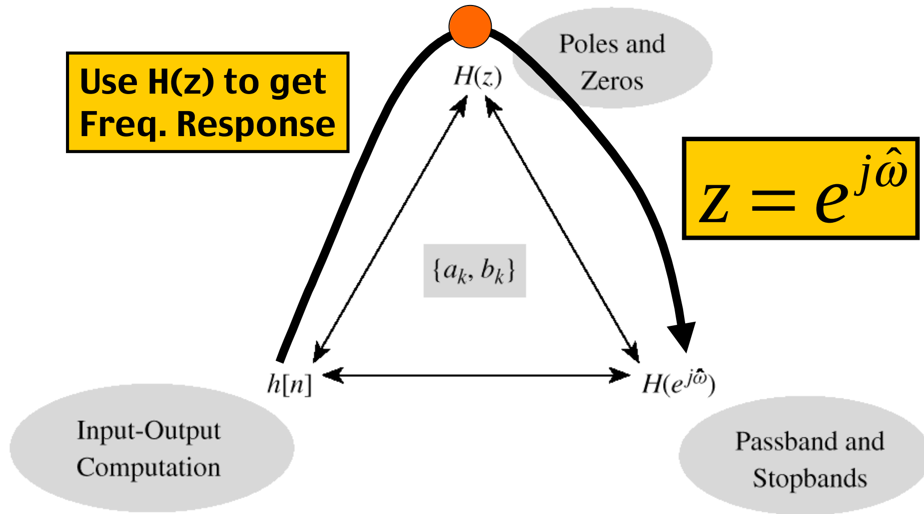


Figure 8.13 Relationship among the n-, z-, and $\hat{\omega}$-domains. The filter coefficients $\{a_k,\ b_k\}$ play a central role.

H(z) = z-Transform{ h[n] }

- **POLYNOMIAL Representation**

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

SYSTEM FUNCTION is Z-Transform of h[n]

- **COMPACT FORM for H(z)**

$$H(z) = \sum_{n=0}^{\infty} ba^n z^{-n} = \sum_{n=0}^{\infty} b(az^{-1})^n$$

$$= \frac{b}{1 - az^{-1}} \quad \text{if } \left|az^{-1}\right| < 1$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{if } |r| < 1$$

First-Order Transform Pair

- **GEOMETRIC SEQUENCE:**

$$h[n] = ba^n u[n] \quad \leftrightarrow \quad H(z) = \frac{b}{1 - az^{-1}}$$

- **USE KNOWN TRANSFORM PAIR:**

$$h[n] = ba^n u[n] = 3(0.8)^n u[n]$$

$$H(z) = \sum_{n} 3(0.8)^n u[n]z^{-n}$$

$$= \sum_{n=0}^{\infty} 3(0.8)^n z^{-n} = \frac{3}{1 - 0.8z^{-1}}$$

Z-Transform of Infinite Length

- **INFINITE–LENGTH Signal**

$$X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

APPLIES to ANY SIGNAL

- **Example:**

$$x[n]=(-0.5)^{n-1}u[n-1]$$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad \text{if } |r|<1$$

$$X(z)=\sum_{n=1}^{\infty}\left((-0.5)^{n-1}u[n-1]\right)z^{-n}$$

$$= z^{-1}-0.5z^{-2}+0.25z^{-3}-\ldots=\frac{z^{-1}}{1+0.5z^{-1}}$$

DELAY PROPERTY of X(z)

- **DELAY in TIME<-->Multiply X(z) by z^{-1}**

$$x[n]\leftrightarrow X(z)$$

$$x[n-1]\leftrightarrow z^{-1}X(z)$$

$$\text{Proof:}\quad \sum_{n=-\infty}^{\infty} x[n-1]z^{-n} = \sum_{\ell=-\infty}^{\infty} x[\ell]z^{-(\ell+1)}$$

$$= z^{-1}\sum_{\ell=-\infty}^{\infty} x[\ell]z^{-\ell} = z^{-1}X(z)$$

Z-Transform of IIR Filter

- **DERIVE the SYSTEM FUNCTION H(z)**
  - Use **DELAY PROPERTY**

$$y[n]=a_1y[n-1]+b_0x[n]+b_1x[n-1]$$

$$Y(z)=a_1z^{-1}Y(z)+b_0X(z)+b_1z^{-1}X(z)$$

EASIER with DELAY PROPERTY

Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n-n_0] \quad \Longleftrightarrow \quad z^{-n_0}X(z)$$

SYSTEM FUNCTION of IIR

- **NOTE the FILTER COEFFICIENTS**

$$Y(z)-a_1z^{-1}Y(z)=b_0X(z)+b_1z^{-1}X(z)$$

$$(1-a_1z^{-1})Y(z)=(b_0+b_1z^{-1})X(z)$$

$$H(z)=\frac{Y(z)}{X(z)}=\frac{b_0+b_1z^{-1}}{1-a_1z^{-1}}=\frac{B(z)}{A(z)}$$

SYSTEM FUNCTION

- DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]$$

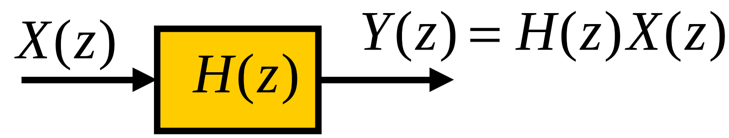
- READ the FILTER COEFFS:

H(z)

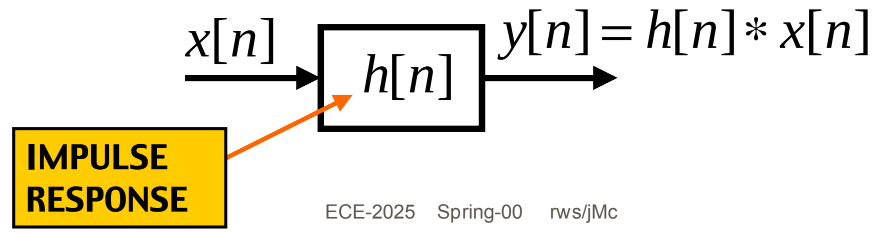
$$Y(z) = \left(\frac{3 - 2z^{-1}}{1 - 0.8z^{-1}} \right) X(z)$$

CONVOLUTION PROPERTY

- MULTIPLICATION of z–TRANSFORMS



- CONVOLUTION in TIME–DOMAIN



POLES & ZEROS

- ROOTS of NUMERATOR & DENOMINATOR

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} \rightarrow H(z) = \frac{b_0 z + b_1}{z - a_1}$$

$$b_0 z + b_1 = 0 \quad \Rightarrow z = -\frac{b_1}{b_0}$$

ZERO: H(z)=0

$$z - a_1 = 0 \quad \Rightarrow z = a_1$$

POLE: H(z) –> ∞

EXAMPLE: Poles & Zeros

- VALUE of H(z) at POLES is INFINITE

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

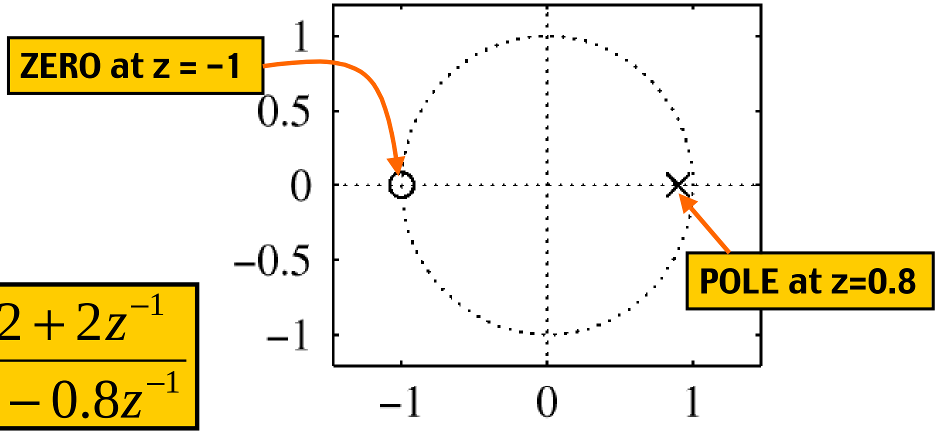
$$H(-1) = \frac{2 + 2(-1)}{1 - 0.8(-1)} = 0$$

ZERO at z=–1

$$H(\tfrac{4}{5}) = \frac{2 + 2(\frac{5}{4})}{1 - 0.8(\frac{5}{4})} = \frac{\frac{7}{5}}{0} \rightarrow \infty$$

POLE at z=0.8

POLE-ZERO PLOT



$$\frac{2+2z^{-1}}{1-0.8z^{-1}}$$

FREQUENCY RESPONSE

- SYSTEM FUNCTION: H(z)
- H(z) has DENOMINATOR
- FREQUENCY RESPONSE of IIR
  - We have H(z)

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

- THREE–DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

FREQUENCY RESPONSE

- EVALUATE on the UNIT CIRCLE

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

$$H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}} = \frac{b_0 + b_1 e^{-j\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}}}$$

FREQ. RESPONSE FORMULA

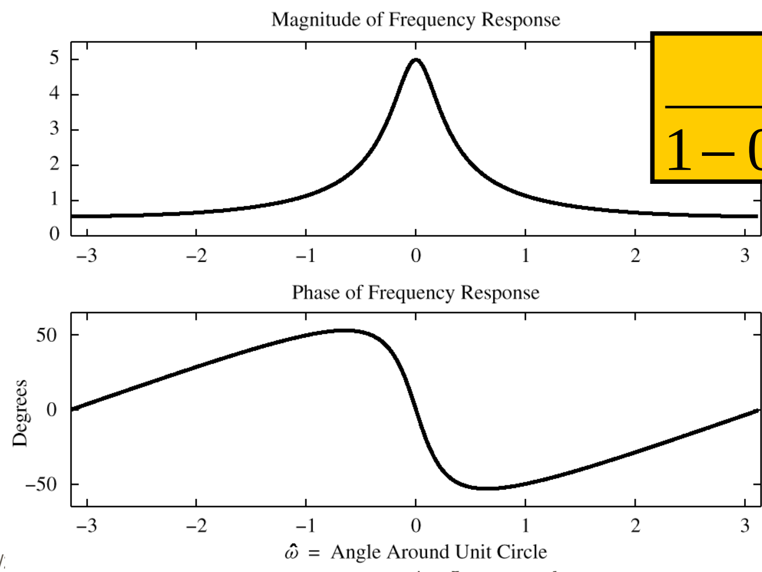
$$H(z) = \frac{1}{1-0.8z^{-1}} \rightarrow H(e^{j\hat{\omega}}) = \frac{1}{1-0.8e^{-j\hat{\omega}}}$$

$$\left|H(e^{j\hat{\omega}})\right|^2 = \left|\frac{1}{1-0.8e^{-j\hat{\omega}}}\right|^2 = \frac{1}{1-0.8e^{-j\hat{\omega}}} \cdot \frac{1}{1-0.8e^{j\hat{\omega}}}$$

$$\frac{1}{1+0.64-0.8e^{-j\hat{\omega}}-0.8e^{j\hat{\omega}}} = \frac{1}{1.64-1.6\cos\hat{\omega}}$$

$$@\hat{\omega}=0, \quad \left|H(e^{j\hat{\omega}})\right|^2 = \frac{1}{0.04} = 25 \qquad @\hat{\omega}=\pi\,?$$

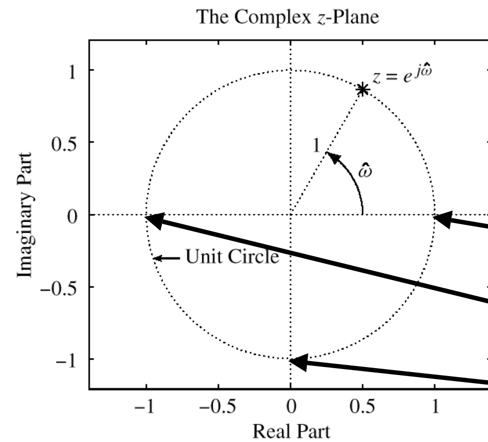
FREQ. RESPONSE from H(z)



$$\frac{1}{1-0.8e^{-j\hat{\omega}}}$$

UNIT CIRCLE

- **MAPPING BETWEEN** z and $\hat{\omega}$



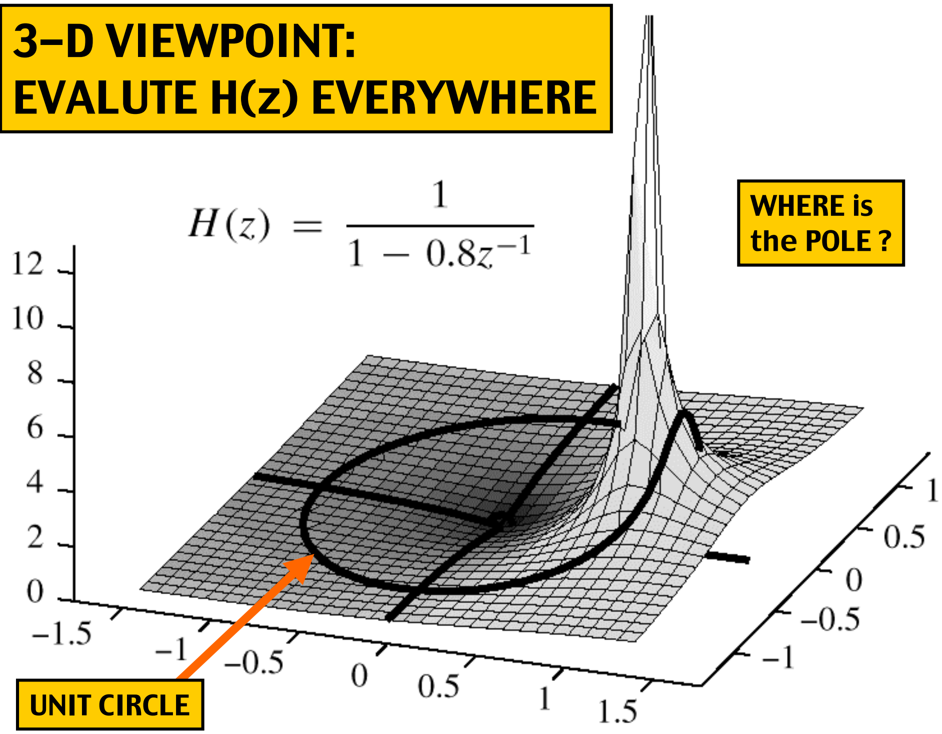
$$z = e^{j\hat{\omega}}$$

$z = 1 \leftrightarrow \hat{\omega} = 0$

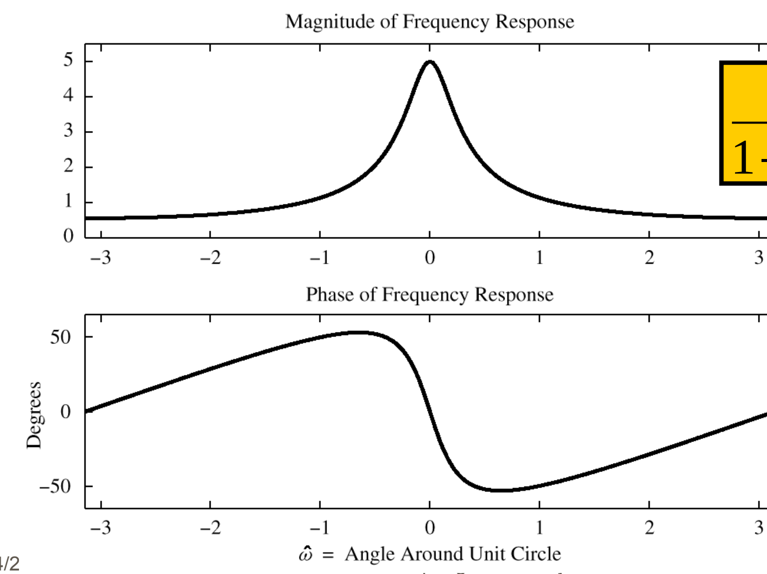
$z = -1 \leftrightarrow \hat{\omega} = \pm\pi$

$z = \pm j \leftrightarrow \hat{\omega} = \pm\frac{1}{2}\pi$

3–D VIEWPOINT:
EVALUTE H(z) EVERYWHERE



FREQ. RESPONSE from 3-D



$$\frac{1}{1-0.8z^{-1}}$$

THREE DOMAINS

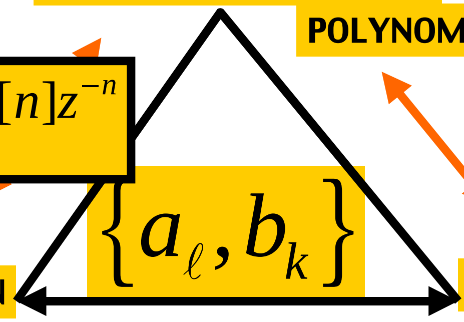
Use H(z) to get Freq. Response

$$H(z) = \sum_{n} h[n] z^{-n}$$

Z-TRANSFORM-DOMAIN

POLYNOMIALS: H(z)

$$z = e^{j\hat{\omega}}$$



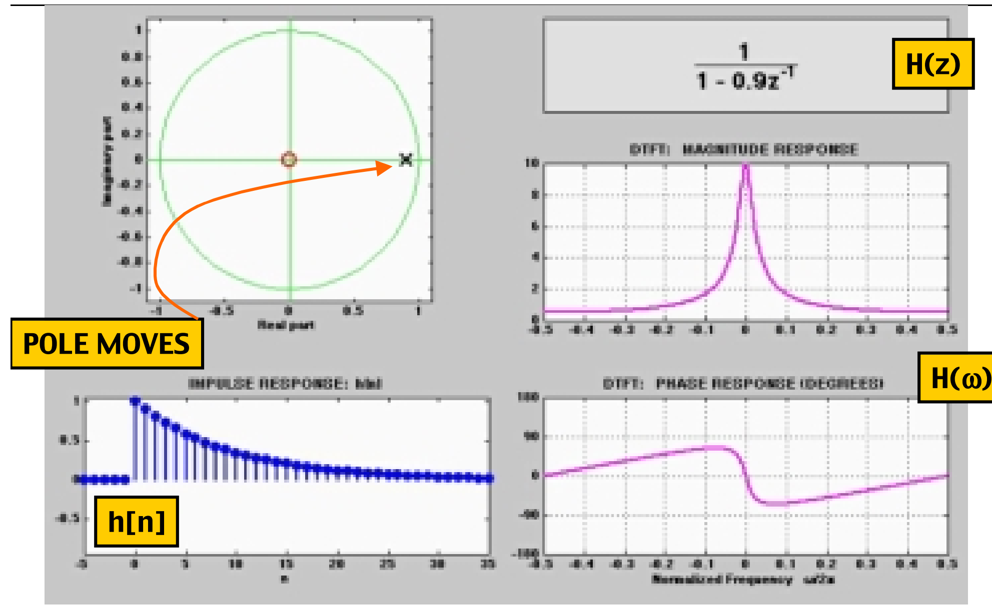
$$\{a_\ell, b_k\}$$

TIME-DOMAIN

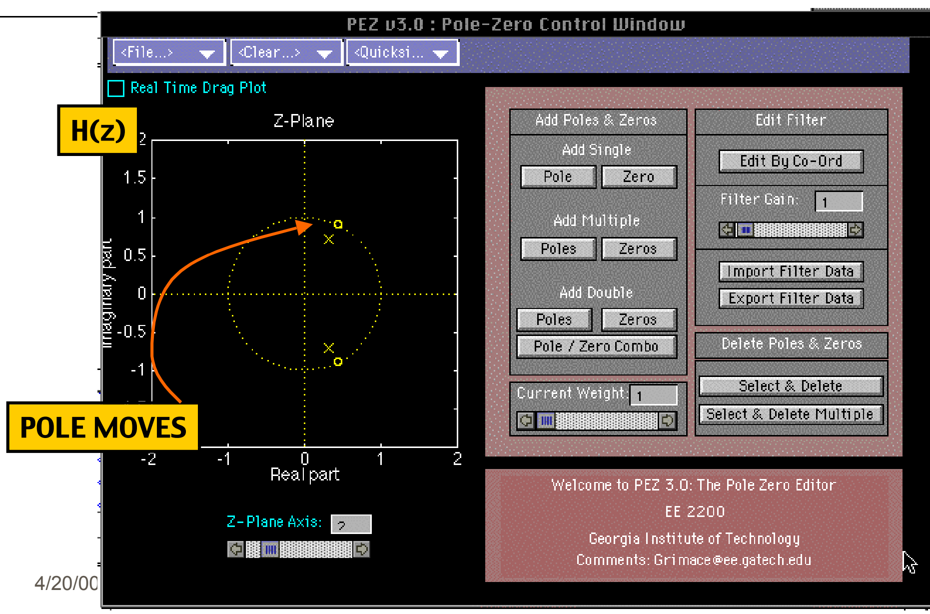
FREQ-DOMAIN

$$y[n] = a_1 y[n-1] + \sum_{k=0}^{M} b_k x[n-k]$$

3 DOMAINS MOVIE: IIR



PeZ GUI for MATLAB: Lab 12



POP QUIZ

- **Given:** $H(z) = \dfrac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$
- **Find the Impulse Response, h[n]**
- **Find the output, y[n]**
  - **When** $x[n] = \cos(0.25\pi n)$

POP QUIZ: Invert Z

- **Given:**

$$H(z)=\frac{2+2z^{-1}}{1-0.8z^{-1}}=\frac{2}{1-0.8z^{-1}}+\frac{2z^{-1}}{1-0.8z^{-1}}$$

- **Find the Impulse Response, h[n]**
  - **Use the DELAY PROPERTY**

$$h[n]=2(0.8)^n u[n]+2(0.8)^{n-1}u[n-1]$$

SINUSOIDAL RESPONSE

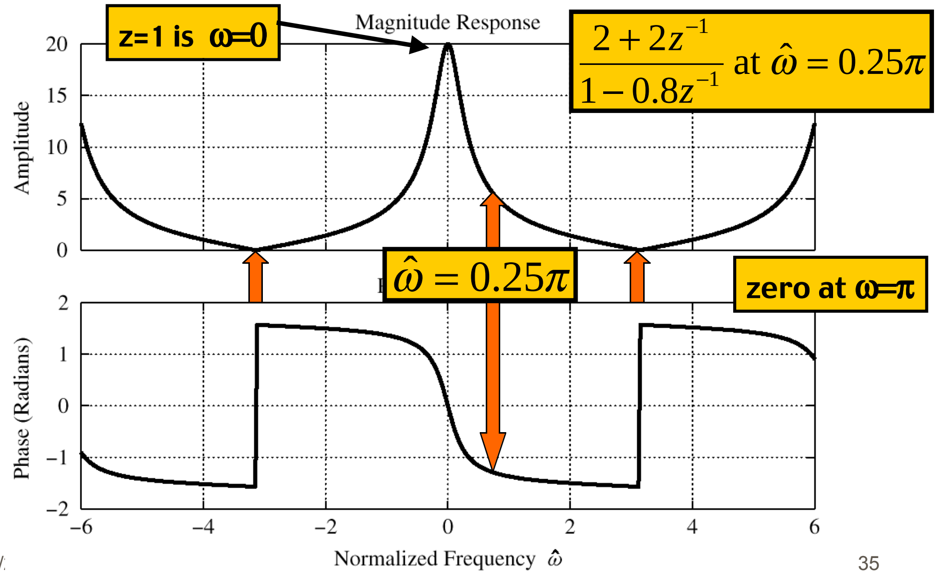
- **x[n] = SINUSOID => y[n] is SINUSOID**
- **Get MAGNITUDE & PHASE from H(z)**

$$\text{if } x[n]=e^{j\hat{\omega}n}\text{, then}$$

$$y[n]=\mathcal{H}(\hat{\omega})e^{j\hat{\omega}n}$$

$$\mathcal{H}(\hat{\omega})=H(e^{j\hat{\omega}})=H(z)\big|_{z=e^{j\hat{\omega}}}$$

Evaluate FREQ. RESPONSE



POP QUIZ: Eval Freq. Resp.

- **Given:** $H(z)=\dfrac{2+2z^{-1}}{1-0.8z^{-1}}$
- **Find output, y[n], when** $x[n]=\cos(0.25\pi n)$
  - **Evaluate at** $z=e^{j0.25\pi}$

$$H(z)=\frac{2+2(\frac{\sqrt{2}}{2}+j\frac{\sqrt{2}}{2})}{1-0.8e^{-j0.25\pi}}=5.182e^{-j1.309}$$

$$y[n]=5.182\cos(0.25\pi n-0.417\pi)$$