# EE-2025 Spring-2000

**Lecture 25**

**3-Domains for IIR**

**24-April-00**

## Info: Web-CT, Lab, HW

- Final Exam, 2–May @ 11:30 AM
  - Calculator, 1 page Handwritten Notes
  - Review: Monday, 1–May @ 6:45 pm
- HW #13 due last day (in Lecture)
- Lab #12 on Pole–Zero Editor (PeZ)
- Labs DEADLINE
  - ALL Lab Reports due by 28–April, no later than Friday at 5 pm.



## READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, pp. 279–300
- Other Reading:
  - Recitation: Ch. 8, pp. 261–272
    - POLES & ZEROS
  - Next Lecture: Chapter 8, all



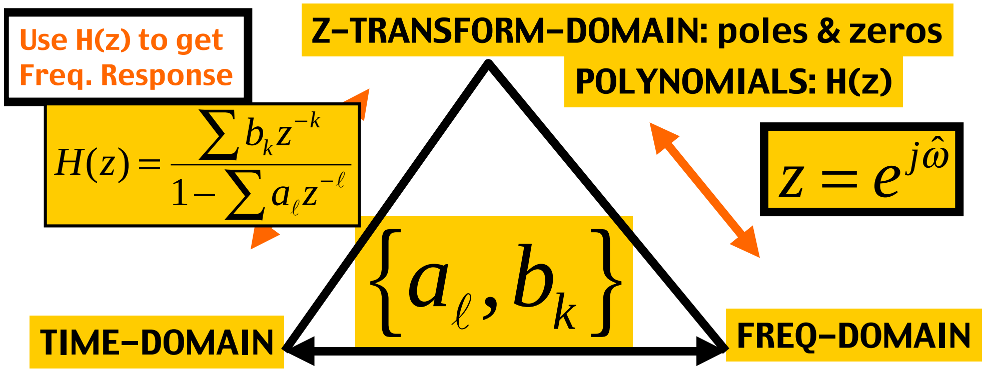
## LECTURE OBJECTIVES

- SECOND–ORDER IIR FILTERS
  - TWO FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]$$

- H(z) can have COMPLEX POLES & ZEROS
- THREE–DOMAIN APPROACH
  - BPFs have POLES NEAR THE UNIT CIRCLE

# THREE DOMAINS




$$y[n] = a_1 y[n-1] + \sum_{k=0}^{M} b_k x[n-k]$$



# Z-TRANSFORM TABLES

**SHORT TABLE OF $z$-TRANSFORMS**

| | $x[n]$ | $\Longleftrightarrow$ | $X(z)$ |
|---|---|---|---|
| 1. | $ax_1[n] + bx_2[n]$ | $\Longleftrightarrow$ | $aX_1(z) + bX_2(z)$ |
| 2. | $x[n - n_0]$ | $\Longleftrightarrow$ | $z^{-n_0}X(z)$ |
| 3. | $y[n] = x[n] * h[n]$ | $\Longleftrightarrow$ | $Y(z) = H(z)X(z)$ |
| 4. | $\delta[n]$ | $\Longleftrightarrow$ | $1$ |
| 5. | $\delta[n - n_0]$ | $\Longleftrightarrow$ | $z^{-n_0}$ |
| 6. | $a^n u[n]$ | $\Longleftrightarrow$ | $\frac{1}{1 - az^{-1}}$ |

# POP QUIZ

- Given: $H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$
- Find the Impulse Response, h[n]
- Find the output, y[n]
  - When $x[n] = \cos(0.25\pi n)$



# SECOND-ORDER FILTERS

- Two FEEDBACK TERMS

**SECOND-ORDER FILTERS**

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$Y(z) = a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z)$$

$$Y(z) - a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z)$$

$$(1 - a_1 z^{-1} - a_2 z^{-2}) Y(z) = (b_0 + b_1 z^{-1} + b_2 z^{-2}) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

# MORE POLES

- Denominator is QUADRATIC
  - 2 Poles: REAL
  - or COMPLEX CONJUGATES

$$\frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 - a_1 z - a_2}$$

**PROPERTY OF REAL POLYNOMIALS**

*A polynomial of degree N has N roots. If all the coefficients of the polynomial are real, the roots either must be real, or must occur in complex conjugate pairs.*



# TWO COMPLEX POLES

- Find Impulse Response ?
  - Can OSCILLATE vs. n
  - "RESONANCE"

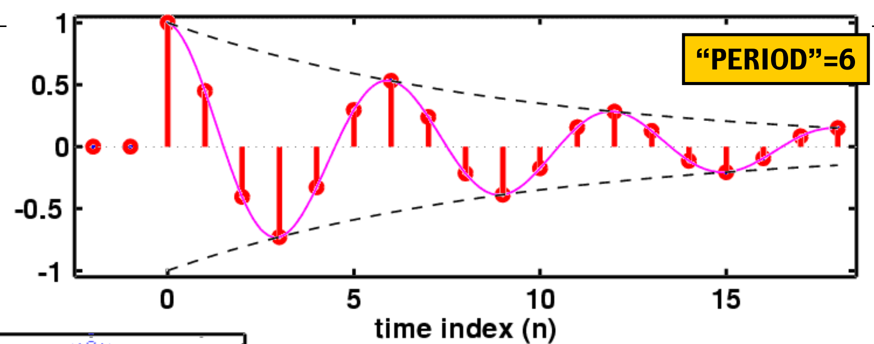
$$(p_k)^n = (re^{j\theta})^n = r^n e^{jn\theta}$$

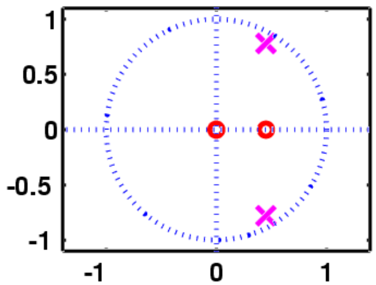
- Find FREQUENCY RESPONSE
  - Depends on Pole Location
  - Close to the Unit Circle?
    - Make BANDPASS FILTER

$$\text{pole} = re^{j\theta}$$
$$r \rightarrow 1\,?$$



# h[n]: Decays & Oscillates




$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3} n) u[n]$$

$$\frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$



# 2nd ORDER EXAMPLE

$$h[n] = 0.9^n \cos(\tfrac{\pi}{3} n) u[n] = 0.9^n \tfrac{1}{2}\left(e^{j\pi n/3} + e^{-j\pi n/3}\right) u[n]$$

$$H(z) = \frac{0.5}{1 - 0.9 e^{j\pi/3} z^{-1}} + \frac{0.5}{1 - 0.9 e^{-j\pi/3} z^{-1}}$$

$$H(z) = \frac{1 - 0.9 \cos(\frac{\pi}{3}) z^{-1}}{(1 - 0.9 e^{j\pi/3} z^{-1})(1 - 0.9 e^{-j\pi/3} z^{-1})}$$

$$H(z) = \frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

# 2nd ORDER Z-transform PAIR

GENERAL ENTRY for z-Transform TABLE

$$h[n] = r^n \cos(\theta n) u[n]$$

$$H(z) = \frac{1 - r\cos\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

$$h[n] = A r^n \cos(\theta n + \varphi) u[n]$$

$$H(z) = A\frac{\cos\varphi - r\cos(\theta - \varphi) z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$



# 2nd ORDER EX: n-Domain

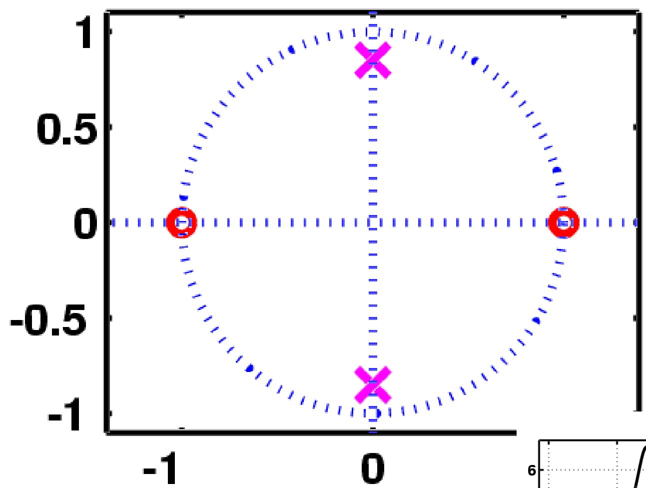
$$\frac{1 - 0.45z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

$$y[n] = 0.9y[n-1] - 0.81y[n-2] + x[n] - 0.45x[n-1]$$

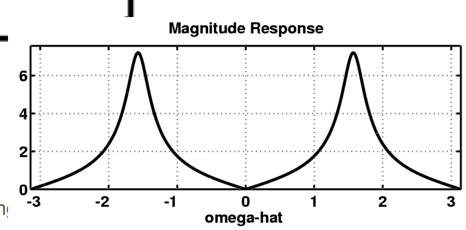
```
aa = [ 1, -0.9, 0.81 ];
bb = [ 1, -0.45 ];
nn = -2:19;
hh = filter( bb, aa, (nn==0) );
HH = freqz( bb, aa, [-pi,pi/100:pi] );
```



# Complex POLE-ZERO PLOT



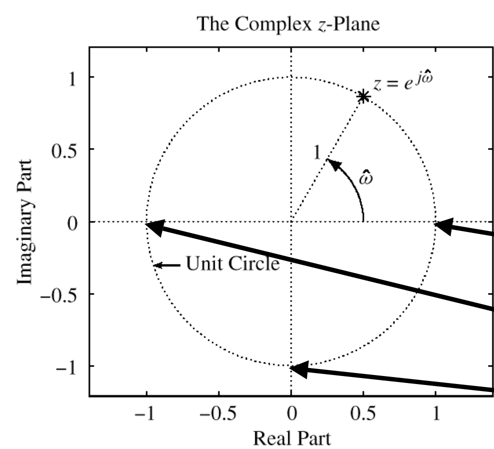

$$\frac{1 - z^{-2}}{1 + 0.7225z^{-2}}$$






# UNIT CIRCLE

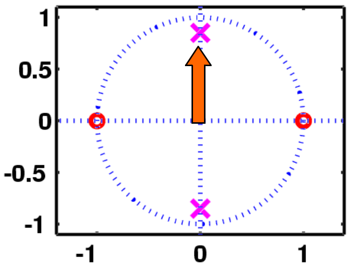
- **MAPPING BETWEEN** $z$ and $\hat{\omega}$



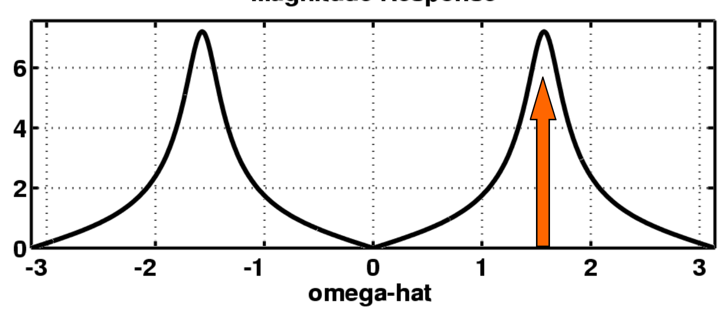

$$z = e^{j\hat{\omega}}$$

| | | |
|---|---|---|
| $z = 1$ | $\leftrightarrow$ | $\hat{\omega} = 0$ |
| $z = -1$ | $\leftrightarrow$ | $\hat{\omega} = \pm\pi$ |
| $z = \pm j$ | $\leftrightarrow$ | $\hat{\omega} = \pm\frac{1}{2}\pi$ |

# FREQUENCY RESPONSE from POLE-ZERO PLOT




$$H(e^{j\hat{\omega}}) = \frac{1 - e^{-j2\hat{\omega}}}{1 + 0.7225e^{-j2\hat{\omega}}}$$

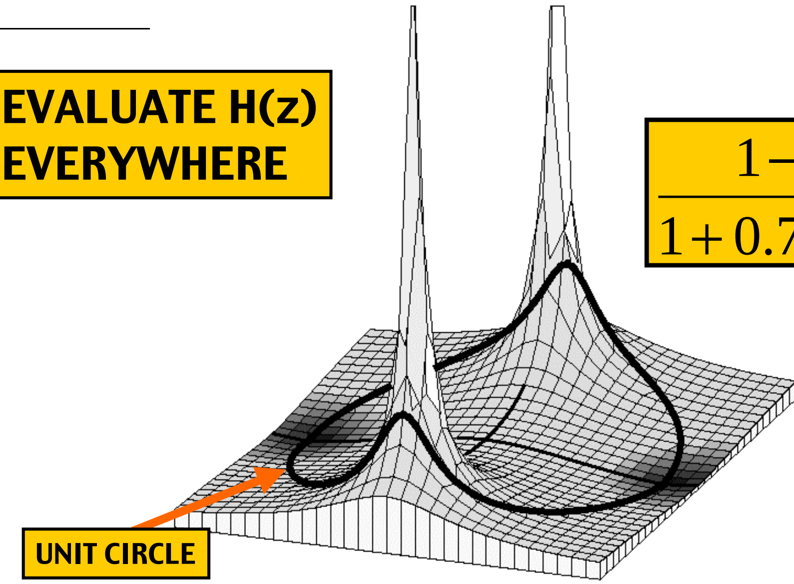





# 3-D VIEW

**EVALUATE H(z) EVERYWHERE**

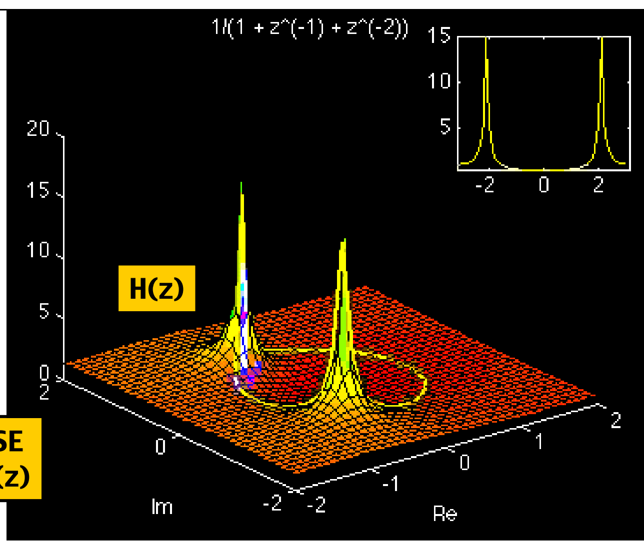
$$\frac{1 - z^{-2}}{1 + 0.7225z^{-2}}$$




The poles are at $z = 0.85e^{\pm j\pi/2}$ and the zeros at $z = \pm 1$.



# FLYING THRU Z-PLANE



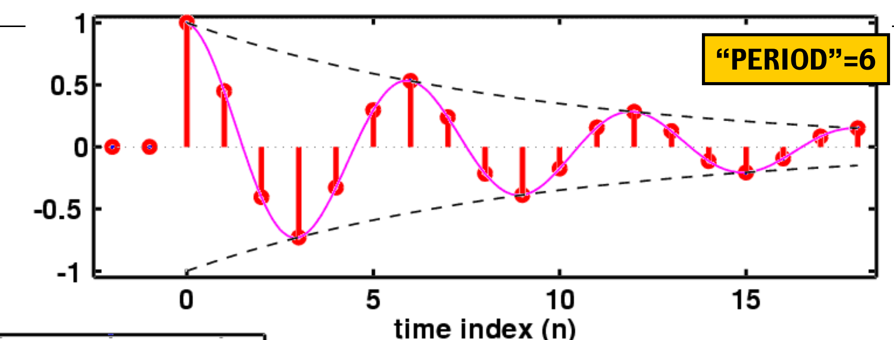

**H(z)**

**H(ω)**

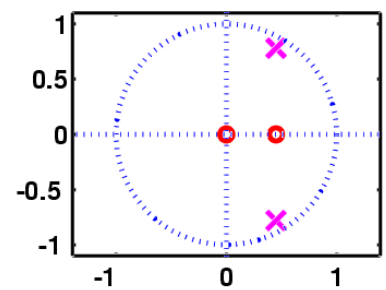
**POLES CAUSE PEAKS in H(z)**



# h[n]: Decays & Oscillates



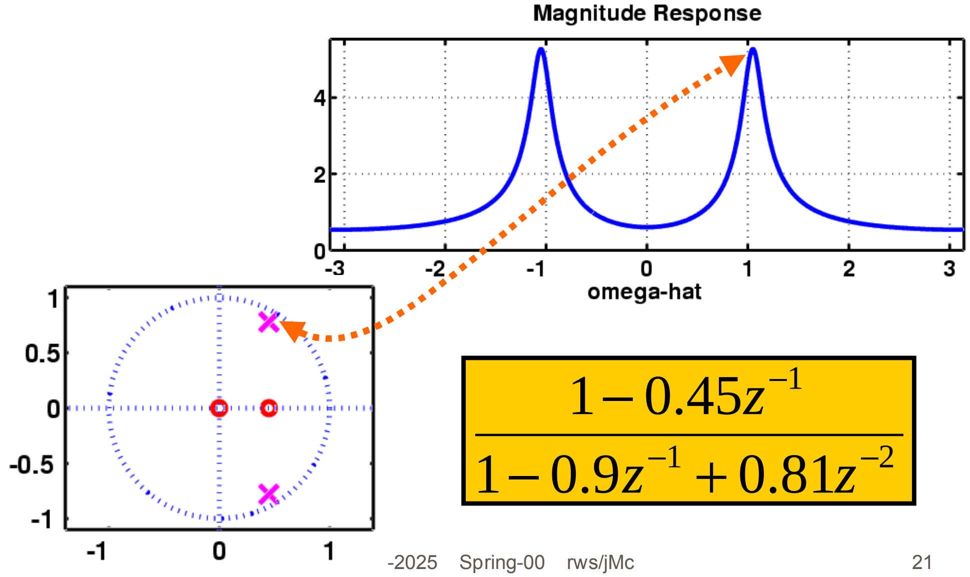

**"PERIOD"=6**




$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3} n) u[n]$$

$$\frac{1 - 0.45z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

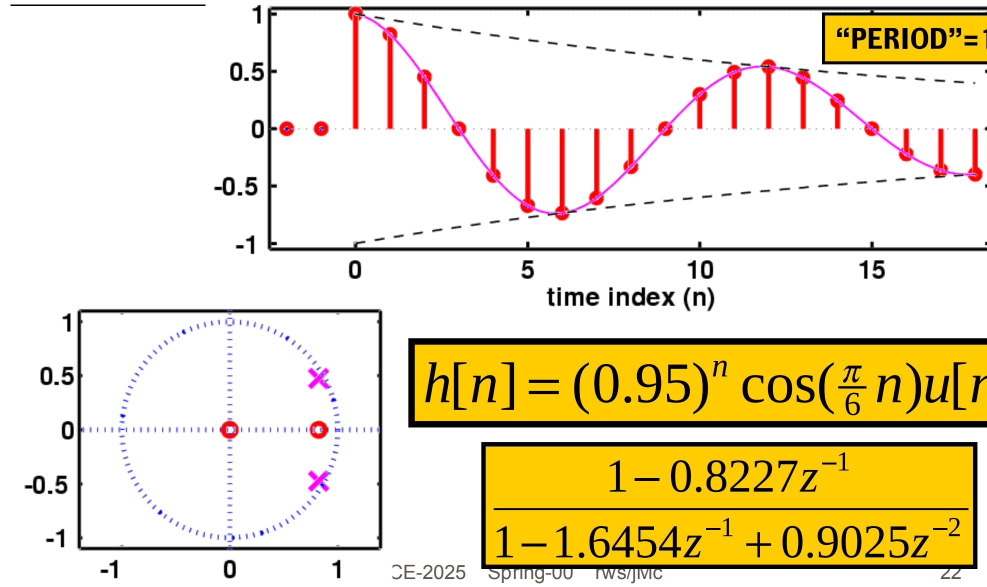
# Complex POLE-ZERO PLOT




$$\frac{1-0.45z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$$



# h[n]: Decays & Oscillates

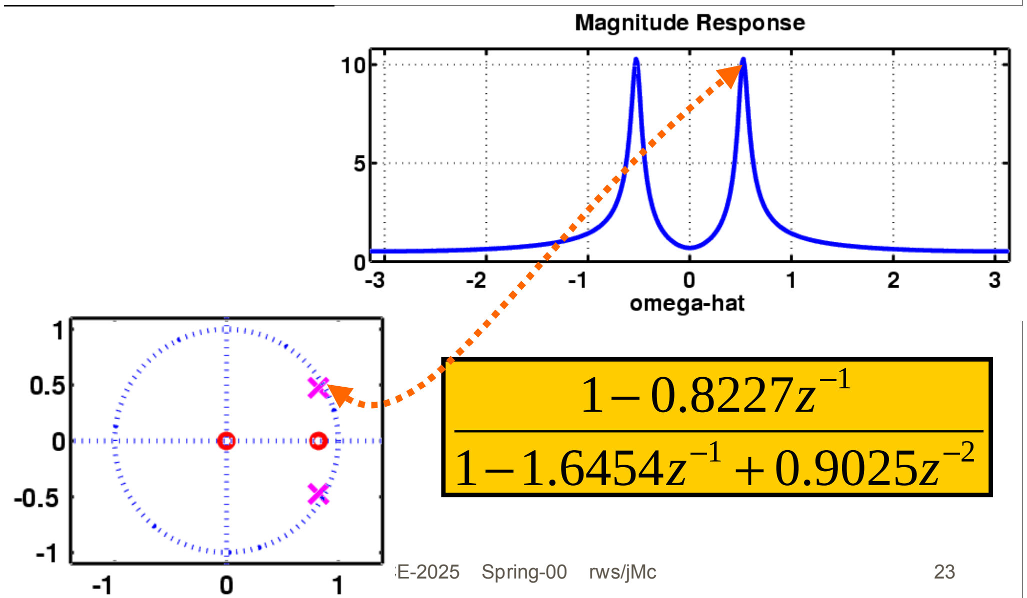



$$h[n]=(0.95)^n\cos(\tfrac{\pi}{6}n)u[n]$$

$$\frac{1-0.8227z^{-1}}{1-1.6454z^{-1}+0.9025z^{-2}}$$



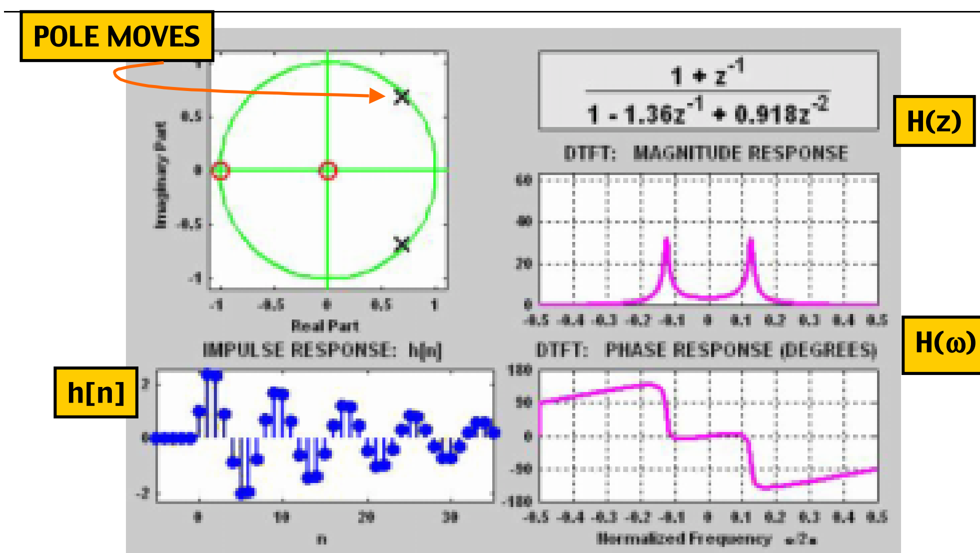
# Complex POLE-ZERO PLOT




$$\frac{1-0.8227z^{-1}}{1-1.6454z^{-1}+0.9025z^{-2}}$$



# 3 DOMAINS MOVIE: IIR



# THREE INPUTS

- **Given:** $\rightarrow H(z) = \dfrac{5}{1+0.8z^{-1}} \rightarrow$
- **Find the output, y[n]**
  - **When**

$$x[n] = \cos(0.2\pi n)$$

$$x[n] = a^n u[n]$$

$$x[n] = \cos(0.2\pi n)u[n]$$



# SINUSOID ANSWER

- **Given:** $H(z) = \dfrac{5}{1+0.8z^{-1}}$
- **The input:** $x[n] = \cos(0.2\pi n)$
- **Then y[n]** $y[n] = M\cos(0.2\pi n + \psi)$

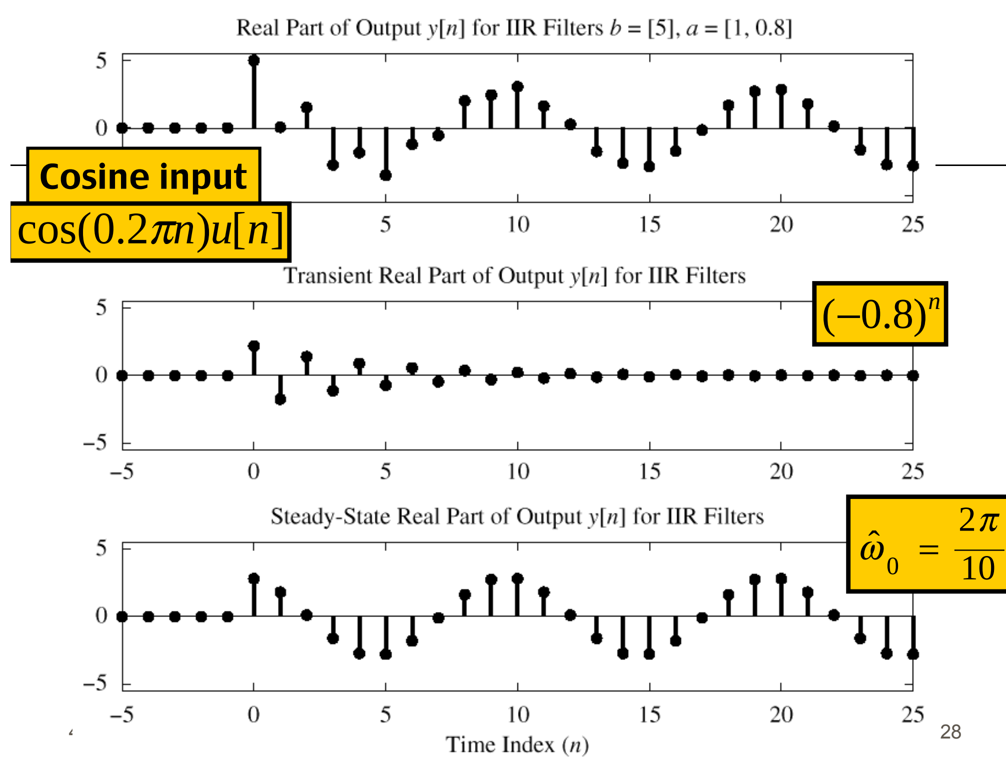
$$H(e^{j0.25\pi}) = \frac{5}{1+0.8e^{-j0.2\pi}} = 2.919e^{j0.089\pi}$$



# SINUSOID starting at n=0

- **We'll look at an example in MATLAB**
  - **cos(0.2πn)**
  - **Pole at –0.8, so $a^n$ is $(-0.8)^n$**
- **There are two components:**
  - **TRANSIENT**
    - **Start-up region just after n=0;** $(-0.8)^n$
  - **STEADY-STATE**
    - **Eventually, y[n] looks sinusoidal.**
    - **Magnitude & Phase from Frequency Response**





# STABILITY

- When Does the TRANSIENT DIE OUT ?

**STEADY-STATE RESPONSE AND STABILITY**

A stable system is one that does not "blow up." This intuitive statement can be formalized by saying that the output of a stable system can always be bounded ($|y[n]| < M_y$) whenever the input is bounded ($|x[n]| < M_x$).[3]

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

$$h[n] = b_0 a_1^n u[n]$$

need $|a_1| < 1$



# STABILITY CONDITION

- ALL POLES INSIDE the UNIT CIRCLE
- UNSTABLE EXAMPLE:

POLE @ z=1.1

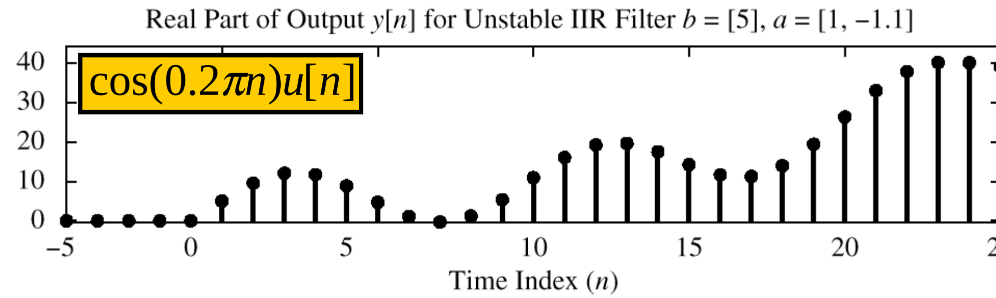



**Figure 8.15** Illustration of an unstable IIR system. Pole is at $z = 1.1$.



# CALCULATE the RESPONSE

$$x[n] = e^{j\hat{\omega}_0 n} u[n]$$

$$X(z) = \frac{1}{1 - e^{j\hat{\omega}_0} z^{-1}}$$

$H(z)$

$$Y(z) = H(z)X(z) = \left(\frac{b_0}{1 - a_1 z^{-1}}\right)\left(\frac{1}{1 - e^{j\hat{\omega}_0} z^{-1}}\right)$$

Use the Z-Transform Method
And PARTIAL FRACTIONS



# GENERAL INVERSE Z

**PROCEDURE FOR INVERSE $z$-TRANSFORMATION ($M < N$)**

1. Factor the denominator polynomial of $H(z)$ and express the pole factors in the form $(1 - p_k z^{-1})$ for $k = 1, 2, \ldots, N$.
2. Make a partial fraction expansion of $H(z)$ into a sum of terms of the form

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} \quad \text{where} \quad A_k = H(z)(1 - p_k z^{-1})\Big|_{z=p_k}$$

3. Write down the answer as

$$h[n] = \sum_{k=1}^{N} A_k (p_k)^n u[n]$$

$(\text{pole})^n$

## SPLIT Y(z) to INVERT

- Need SUM of Terms:

$$Y(z) = H(z)X(z) = \left(\frac{b_0}{1 - a_1 z^{-1}}\right)\left(\frac{1}{1 - e^{j\hat{\omega}_0} z^{-1}}\right)$$

$$= \frac{b_0}{(1 - a_1 z^{-1})(1 - e^{j\hat{\omega}_0} z^{-1})}$$

$$Y(z) = \frac{\left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)}{1 - a_1 z^{-1}} + \frac{\left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right)}{1 - e^{j\hat{\omega}_0} z^{-1}}$$



## INVERT Y(z) to y[n]

- Use the Z-Transform Table

$$Y(z) = \frac{\left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)}{1 - a_1 z^{-1}} + \frac{\left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right)}{1 - e^{j\hat{\omega}_0} z^{-1}}$$

$$y[n] = \left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)(a_1)^n u[n] + \left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right) e^{j\hat{\omega}_0 n} u[n]$$



## TWO PARTS of y[n]

- TRANSIENT
  - Acts Like (pole)$^n$
  - Dies out ?
    - IF $|a_1|<1$

$$\left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right) a_1^n u[n]$$

- STEADY-STATE
  - Depends on the input
  - e.g., Sinusoidal

$$\left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right) e^{j\hat{\omega}_0 n} u[n]$$



## STEADY STATE HAPPENS

- When Transient dies out
- Limit as "n" approaches infinity
- Use Frequency Response to get Magnitude & Phase for sinusoid

$$y_{ss}[n] \rightarrow \left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right) e^{j\hat{\omega}_0 n} = H(e^{j\hat{\omega}_0}) e^{j\hat{\omega}_0 n}$$