EE-2025 Spring 2001

Lecture 6

Time-Varying Frequency and Fourier Series Coefficients

26-January-01

Web-CT Info

- Check the Bulletin Board for msgs
- **Get Notes on Fourier Series**
  - 17 pages, posted to WebCT
  - Replacement for pp. 62-66 in Chapter 3
- Prob Set #3 due next week
- Prob Set #2 solution posted
- Problem Set #4 posted Friday or Saturday

Quiz #1 Info

- Quiz #1 on 2-Feb (Friday)
  - Calculator OK, and one page of notes
  - Coverage: HW #1, #2, and #3
- Old Quizzes & Problems are linked via WebCT: Resources

Lab Info

- Lab #2 Report
  - Turn in during your lab time
  - Write-up section 4 on "multipath"
  - Finish INSTRUCTOR VERIFICATION in Lab
  - ERRATA ? ALWAYS Check Bulletin Board
- Lab #3 will be posted Friday or Saturday
  - Learn some Music Notation

LECTURE

Time-Varying Frequency

- Frequency can change vs. time
  - Continuously, not stepped
- **FREQUENCY MODULATION (FM)**

$$x(t) = \cos(2\pi f_c t + v(t))$$

VOICE

- CHIRP SIGNALS
  - Linear Frequency Modulation (LFM)

New Signal: Linear FM

- Called Chirp Signals (LFM)
  - Quadratic phase

QUADRATIC

$$x(t) = A\cos(\alpha t^2 + 2\pi f_0 t + \varphi)$$

- Freq will change LINEARLY vs. time
  - Example of Frequency Modulation (FM)
  - Define "instantaneous frequency"

INSTANTANEOUS FREQ

- Definition

$$x(t) = A\cos(\psi(t))$$

$$\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t)$$

Derivative of the "Angle"

- For Sinusoid:

$$x(t) = A\cos(2\pi f_0 t + \varphi)$$

$$\psi(t) = 2\pi f_0 t + \varphi$$

Makes sense

$$\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t) = 2\pi f_0$$

INSTANTANEOUS FREQ of the Chirp

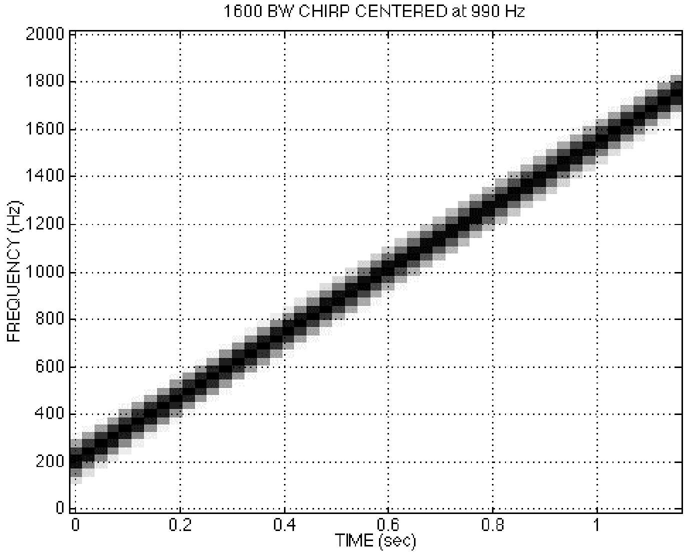
- Chirp Signals have Quadratic phase
- Freq will change LINEARLY vs. time

$$x(t) = A\cos(\alpha t^2 + \beta t + \varphi)$$

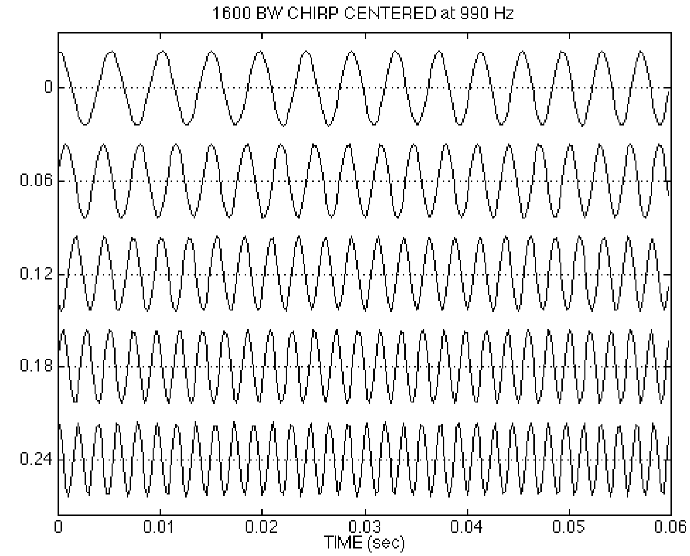
$$\Rightarrow \psi(t) = \alpha t^2 + \beta t + \varphi$$

$$\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t) = 2\alpha t + \beta$$

CHIRP SPECTROGRAM



CHIRP WAVEFORM



OTHER CHIRPS

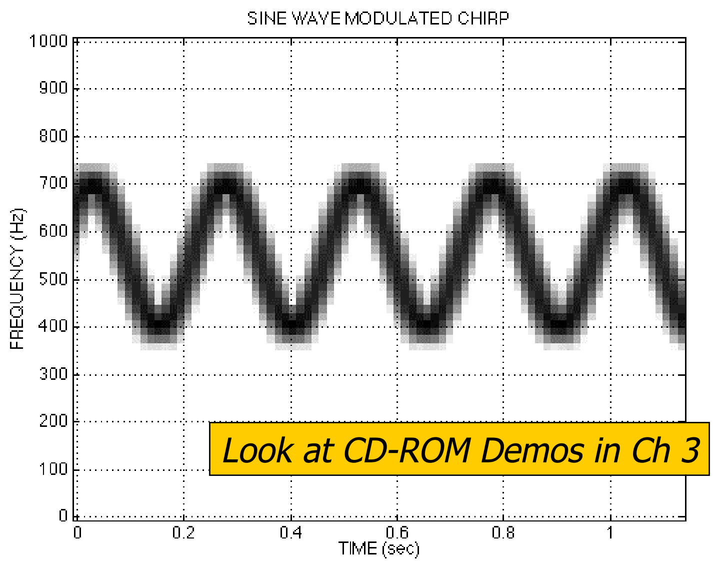
- ψ(t) can be anything:

$$x(t) = A\cos(\alpha\cos(\beta t) + \varphi)$$

$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t) = -\alpha\beta\sin(\beta t)$$

- ψ(t) could be speech or music:
  - FM radio broadcast

SINE-WAVE FREQUENCY MODULATION (FM)



Look at CD-ROM Demos in Ch 3

READING ASSIGNMENTS

- This Lecture:
  - **Notes on Fourier Series**
    - 17 pages, posted to WebCT
    - Replace pp 62-66 in Chapter 3
- Other Reading:
  - Next Lecture: Chap. 4 on Sampling

LECTURE OBJECTIVES

- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi k/T_0)t}dt$$

- **ANALYSIS** via Fourier Series
  - For PERIODIC signals: $x(t+T) = x(t)$
  - **SPECTRUM from the Fourier Series**

HISTORY

- Jean Baptiste Joseph Fourier
  - 1807 thesis (memoir)
    - On the Propagation of Heat in Solid Bodies
  - Heat !
  - Napoleonic era
- http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html

Joseph Fourier

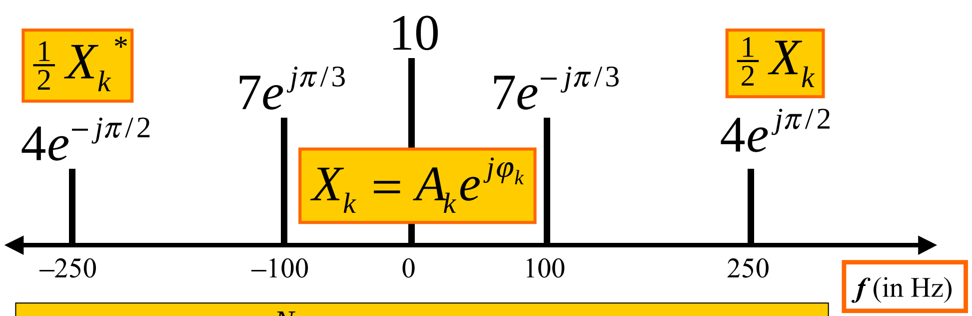
lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Find out more at:
http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fourier.html

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \tfrac{1}{2} X_k e^{j2\pi f_k t} + \tfrac{1}{2} X_k^* e^{-j2\pi f_k t} \right\}$$

Harmonic Signal

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$x(t) = X_0 + \sum_{k=1}^{N} \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \sum_{k=1}^{N} \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$\omega_0 = \frac{2\pi k}{T_0} = \left(\frac{2\pi}{T_0}\right)k = 2\pi(f_0)k$$

Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

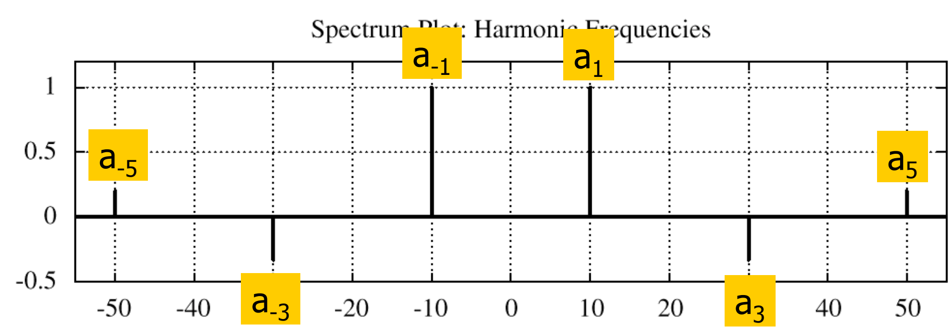
$$a_k = \tfrac{1}{2} X_k = \tfrac{1}{2} A_k e^{j\varphi_k}$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$X_k = A_k e^{j\varphi_k}$$

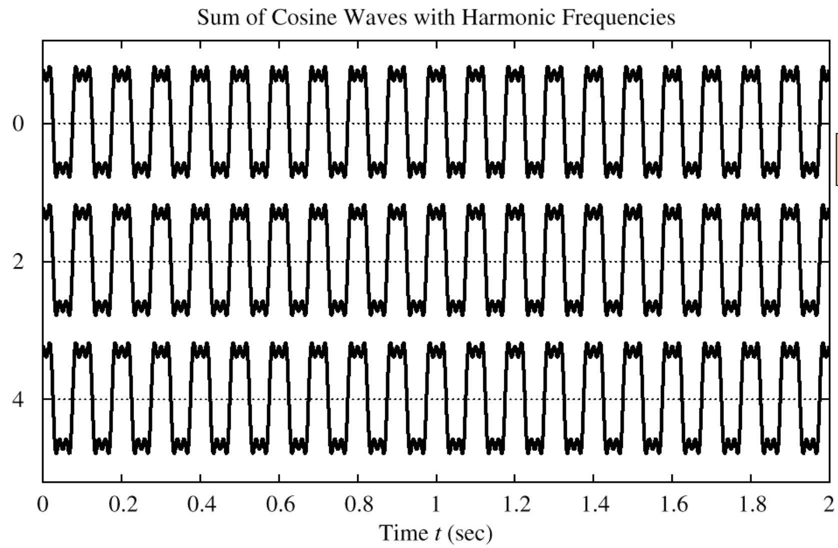
COMPLEX AMPLITUDE

Harmonic Signal (3 Freqs)



a_k is the complex amplitude for kf_0

Harmonic Signal (3 Freqs)



T=0.1

SYNTHESIS vs. ANALYSIS

- SYNTHESIS
  - Easy
  - Given (ω_k,A_k,ϕ_k) create x(t)
- Synthesis can be HARD
  - Synthesize Speech so that it sounds good
- ANALYSIS
  - Hard
  - Given x(t), extract (ω_k,A_k,ϕ_k)
  - How many?
  - Need algorithm for computer

STRATEGY

- ANALYSIS
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - The answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}dt$$

Fourier Series Integral

- HOW do you determine a_k from x(t) ?

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt$$

FUNDAMENTAL FREQ: $f_0=1/T_0$

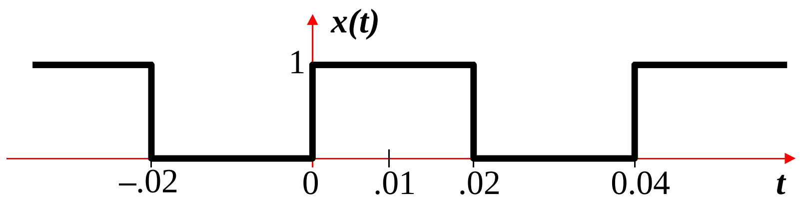
$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt \quad \text{(DC Component)}$$

$$a_{-k} = a_k^* \quad \text{when } x(t) \text{ is real}$$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \tfrac{1}{2}T_0 \\ 0 & \tfrac{1}{2}T_0 \le t < T_0 \end{cases}$$

for $T_0 = 0.04\,\text{sec}$:



FS for a SQUARE WAVE

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}\,dt \qquad (k \ne 0)$$

$$a_k = \frac{1}{.04}\int_0^{.02} 1e^{-j2\pi kt/.04}\,dt = \tfrac{1}{.04(-j\pi k/.02)}\, e^{-j\pi kt/.02}\Big|_0^{.02}$$

$$= \tfrac{1}{(-2j\pi k)}\left(e^{-j\pi k} - 1\right) = \frac{1-(-1)^k}{j2\pi k}$$

DC Coefficient, a_0

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}\,dt \qquad (k = 0)$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0}(\text{AREA})$$

$$a_0 = \tfrac{1}{.04}\int_0^{.02} dt = \tfrac{1}{.04}(.02 - 0) = \tfrac{1}{2}$$

Fourier Coefficients a_k

- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

$$a_k = \frac{1-e^{-j\pi k}}{j2\pi k} = \begin{cases} \dfrac{1}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \tfrac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

$\omega_0 = 2\pi / (0.04) = 2\pi(25)$

$$a_k = \begin{cases} \dfrac{-j}{\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$