EE-2025 Spring 2001

Lecture 6

Fourier Series and Bandlimited Signals

29-January-01

Web-CT Info

- Check the Bulletin Board for msgs
- **Get Notes on Fourier Series**
  - 17 pages, posted to WebCT
  - Replacement for pp. 62-66 in Chapter 3
- Prob Set #3 due this week - solution posted at 6pm on Thursday 1-Feb.
- Problem Set #4 posted. Mon. and Tues. Recitations turn in PS #4 at lab on Weds. and Thurs. next week.

Quiz #1 Info

- Quiz #1 on 2-Feb (Friday)
  - Calculator OK, and one page of notes
  - Coverage: HW #1, #2, and #3
- Old Quizzes & Problems are linked via WebCT: Resources
- ECE2025B and Prof. McClellan's Rec. (L11) report to Room VL-457 for exam
- REVIEW SESSION: Thursday, 1-Feb. 7:30-9:30pm in VanLeer Auditorium. Will be broadcast to GTREP students.

Lab Info

- Lab #2 Report
  - Turn in during your lab time
  - Write-up section 4 on "multipath"
  - Finish INSTRUCTOR VERIFICATION in Lab
  - ERRATA ? ALWAYS Check Bulletin Board
- Lab #3 is posted
  - Learn some Music Notation

LECTURE

My Office Hours

- Monday 1 - 3 pm
- Thursday 4 - 6 pm
- REVIEW SESSION: This Thursday, 1-Feb. 7:30-9:30pm in VanLeer Auditorium. Will be broadcast to GTREP students.

EE-2025 Georgia Tech

Lecture 6a

Fourier Series and Bandlimited Signals

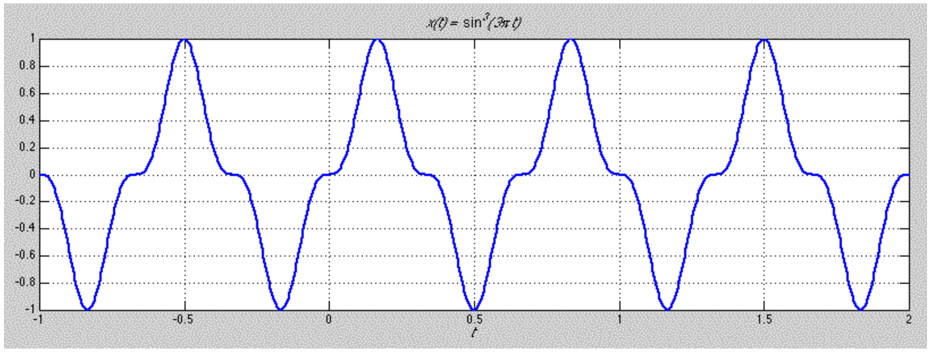
READING ASSIGNMENTS

- This Lecture:
  - **Notes on Fourier Series**
    - 17 pages, posted to WebCT
    - Replace pp 62-66 in Chapter 3
- Other Reading:
  - Next Lecture: Chap. 4 on Sampling

LECTURE OBJECTIVES

- Continue the discussion of Fourier series
- **ANALYSIS** via Fourier Series
  - For PERIODIC signals: $x(t+T_0) = x(t)$
  - **SPECTRUM from the Fourier Series**
- Introduce the concept of a bandlimited signal.

Example $x(t) = \sin^3(3\pi t)$

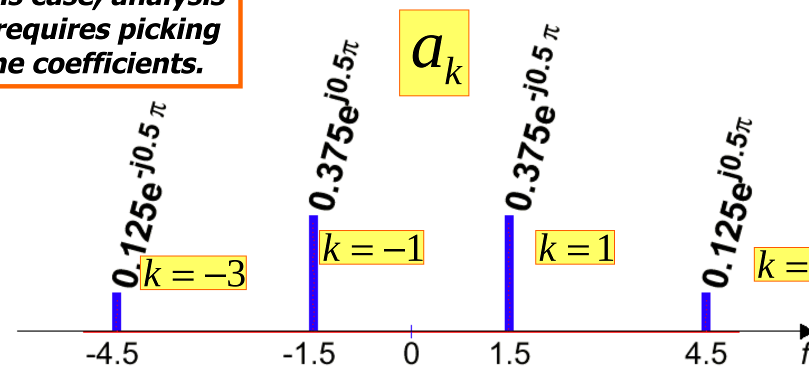


$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

Example $x(t) = \sin^3(3\pi t)$

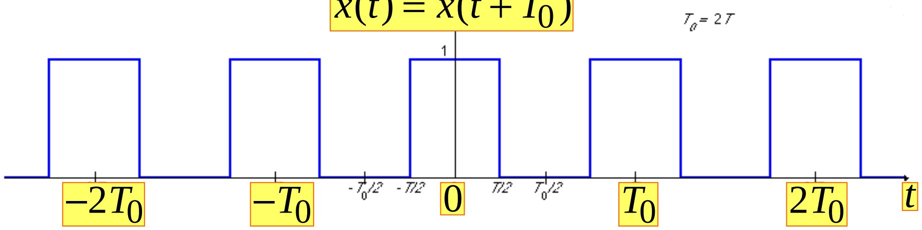
$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

In this case, analysis just requires picking off the coefficients.



General Periodic Signals

$x(t) = x(t + T_0)$



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$ ← ***Fourier Synthesis***

Fundamental Freq. $\omega_0 = 2\pi / T_0 = 2\pi f_0$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}dt$$ ← ***Fourier Analysis***

Fourier Series Integral

- HOW do you determine a_k from x(t) ?

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}dt$$

FUNDAMENTAL FREQ: $f_0 = 1/T_0$ or $\omega_0 = 2\pi/T_0$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt \quad \text{(DC Component)}$$

$$a_{-k} = a_k^* \quad \text{when } x(t) \text{ is real}$$

ORTHOGONALITY of exp(j)

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j2\pi mt/T_0} dt = \frac{T_0}{-j2\pi m} e^{-j2\pi mt/T_0} \Big|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m}\left(e^{-j2\pi m} - 1\right)$$

$$\int_0^{T_0} e^{-jm\omega_0 t} dt = 0 \qquad \omega_0 = \frac{2\pi}{T_0}$$

ORTHOGONALITY of exp(j)

- INTEGRATE over ONE PERIOD

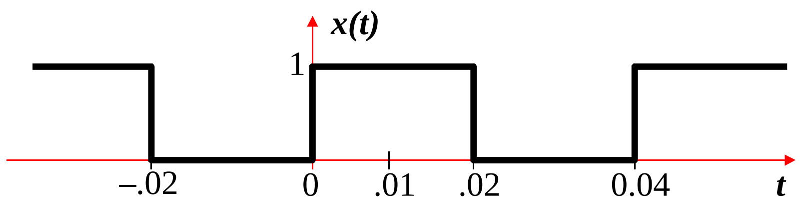
$$\frac{1}{T_0}\int_0^{T_0} e^{j2\pi \ell t/T_0} e^{-j2\pi kt/T_0} dt = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$$

$$\frac{1}{T_0}\int_0^{T_0} e^{j2\pi(\ell - k)t/T_0} dt$$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \tfrac{1}{2}T_0 \\ 0 & \tfrac{1}{2}T_0 \le t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec:



FS for a SQUARE WAVE

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j2\pi kt/T_0} dt \qquad (k \neq 0)$$

$$a_k = \frac{1}{.04}\int_0^{.02} 1 e^{-j2\pi kt/.04} dt = \frac{1}{.04(-j\pi k/.02)} e^{-j\pi kt/.02}\Big|_0^{.02}$$

$$= \frac{1}{(-2j\pi k)}\left(e^{-j\pi k} - 1\right) = \frac{1-(-1)^k}{j2\pi k}$$

DC Coefficient, a_0

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}\,dt \qquad (k=0)$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt = \frac{1}{T_0}(\text{AREA})$$

$$a_0 = \tfrac{1}{.04}\int_0^{.02} dt = \tfrac{1}{.04}(.02-0) = \tfrac{1}{2}$$

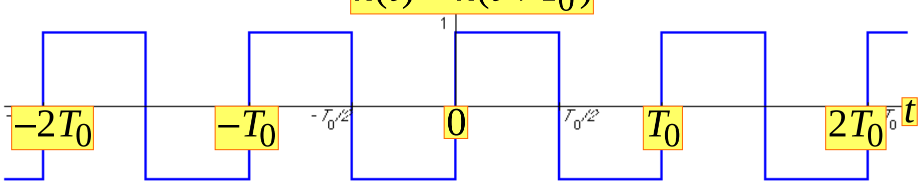
Fourier Coefficients a_k

- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

$$a_k = \frac{1-e^{-j\pi k}}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k=\pm1,\pm3,\ldots \\ 0 & k=\pm2,\pm4,\ldots \\ \frac{1}{2} & k=0 \end{cases}$$

Square Wave Signal

$$x(t) = x(t+T_0)$$



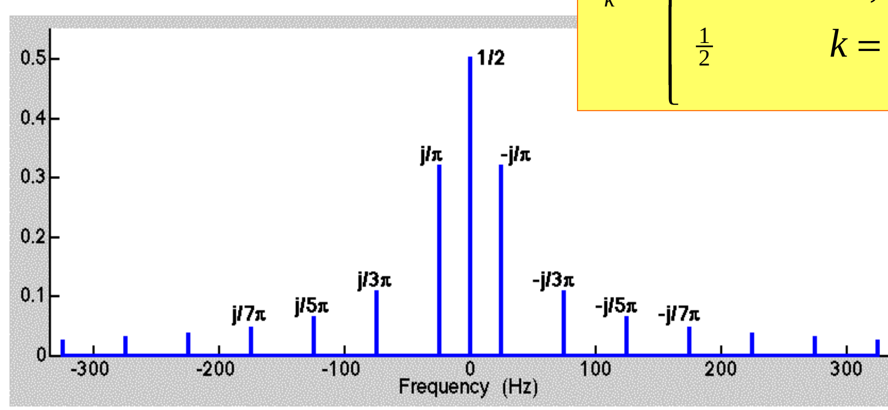
$$a_k = \frac{1}{T_0}\int_0^{T_0/2}(1)e^{-j\omega_0 kt}dt + \frac{1}{T_0}\int_{T_0/2}^{T_0}(-1)e^{-j\omega_0 kt}dt$$

$$a_k = \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_0^{T_0/2} - \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_{T_0/2}^{T_0} = \frac{1-e^{-j\pi k}}{j\pi k}$$

Spectrum from Fourier Series

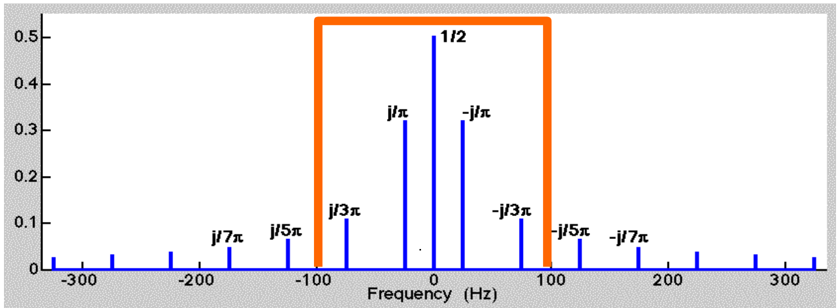
$$\omega_0 = 2\pi/(0.04) = 2\pi(25)$$

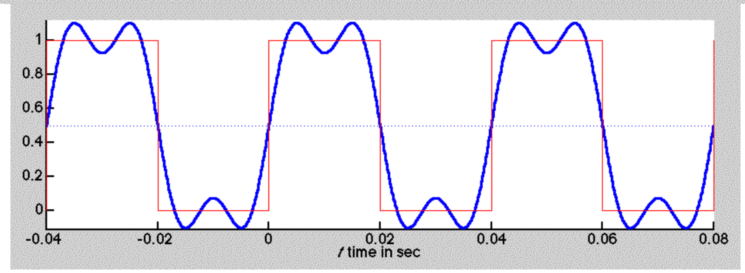
$$a_k = \begin{cases} \frac{-j}{\pi k} & k=\pm1,\pm3,\ldots \\ 0 & k=\pm2,\pm4,\ldots \\ \frac{1}{2} & k=0 \end{cases}$$



Synthesis: 1st & 3rd Harmonics

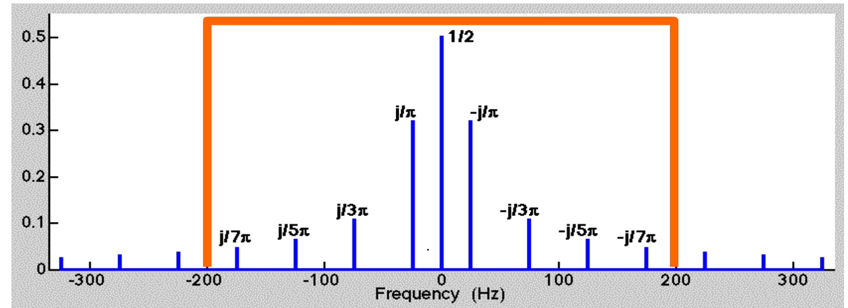
$$y(t)=\frac{1}{2}+\frac{2}{\pi}\cos\left(2\pi(25)t-\tfrac{\pi}{2}\right)+\frac{2}{3\pi}\cos\left(2\pi(75)t-\tfrac{\pi}{2}\right)$$

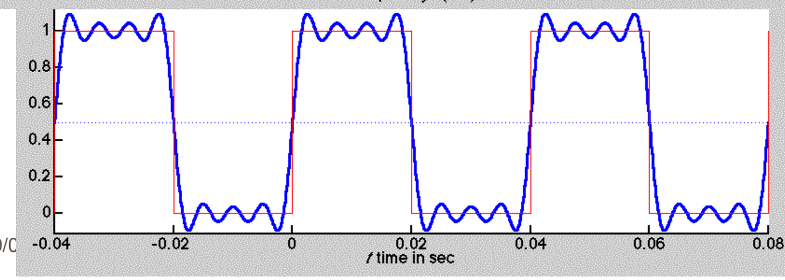




Synthesis: up to 7th Harmonic

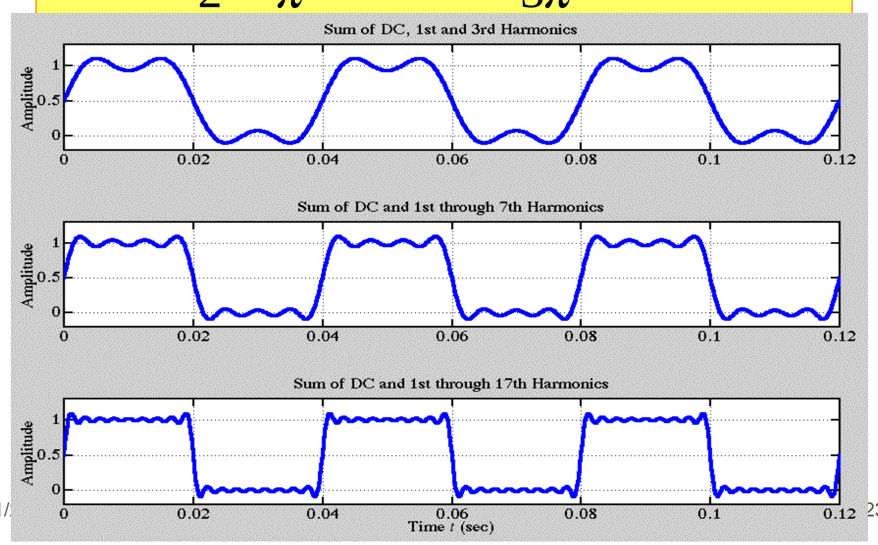
$$y(t)=\frac{1}{2}+\frac{2}{\pi}\cos\left(50\pi t-\tfrac{\pi}{2}\right)+\frac{2}{3\pi}\sin(150\pi t)+\frac{2}{5\pi}\sin(250\pi t)+\frac{2}{7\pi}\sin(350\pi t)$$





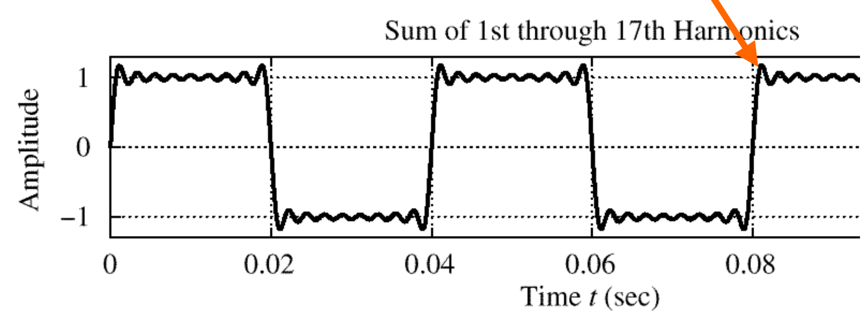
Fourier Synthesis

$$x_N(t)=\frac{1}{2}+\frac{2}{\pi}\sin(\omega_0 t)+\frac{2}{3\pi}\sin(3\omega_0 t)+\ldots$$



Gibbs' Phenomenon

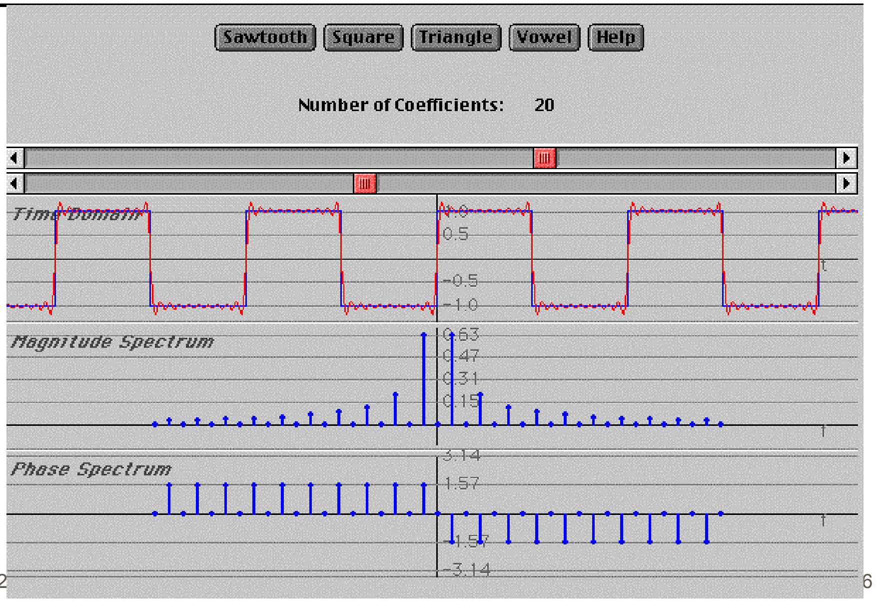
- Convergence at DISCONTINUITY of x(t)
  - There is always an overshoot
  - 9% for the Square Wave case



Fourier Series Demo

- Fourier Series Java Applet
  - Greg Slabaugh
    - Interactive
    - http://users.ece.gatech.edu/~slabaugh/java/fourier/fourier.html

Fourier Series Java Applet



Bandlimited Signals

- A bandlimited signal has all its frequencies below a certain limit ω_N.
  - A square wave is *not* a bandlimited signal since its non-zero spectrum components go all the way up to infinity.
  - Bandlimited signals are very smooth.
  - Bandlimited signals can be sampled and then reconstructed exactly. This is the basis for all of modern communications and signal processing.