EE-2025 Spring-2001

Lecture 7

Sampling & Aliasing

5-Feb-01

Information

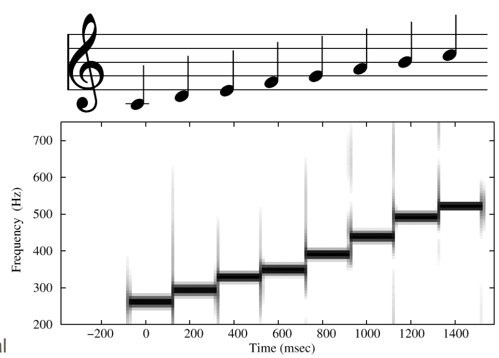
- Quiz Information
  - Will be handed back in Rec. or Lab
- Problem Set #4 due this week. Problem Set#5 is posted
- Lab #4 is Music Synthesis
  - Worth 150 Points
  - Formal lab Report
  - Listening Tests the following week.

CD-ROM DEMOS

- USE THE DEMOS
- Chapter 3: Spectrum

LECTURE

  - DEMOS of SPECTROGRAM
  - BEAT NOTES/AM
  - SPEECH
  - MUSIC
  - FM & Chirps



EE-2025 Georgia Tech

Lecture 7

Sampling & Aliasing

READING ASSIGNMENTS

- This Lecture:
  - Chapter 4, pp. 83-94
- Other Reading:
  - Recitation: Chapter 4, pp. 90-100
    - Strobe Demo
  - Next Lecture: Chap. 4, pp. 100-111

LECTURE OBJECTIVES

- SAMPLING can cause ALIASING
  - Sampling Theorem
  - Sampling Rate > 2(Highest Frequency)
- Spectrum for digital signals, x[n]
  - Normalized Frequency

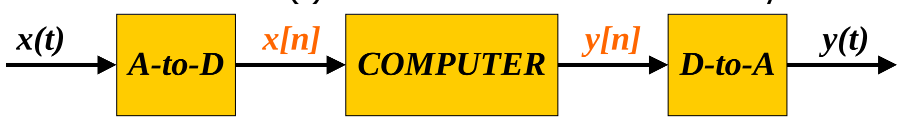
$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi\ell$$

ALIASING

Systems Process (Modify) Signals

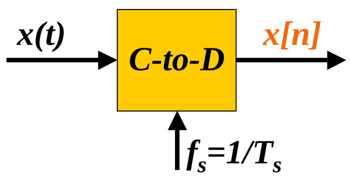
- ANALOG/ELECTRONIC:

  - Circuits: resistors, capacitors, op-amps
  - Improve x(t), e.g., image deblurring
  - Extract Information from x(t)
- DIGITAL/MICROPROCESSOR
  - Convert x(t) to numbers stored in memory



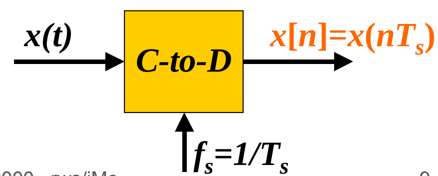
SAMPLING $x(t)$

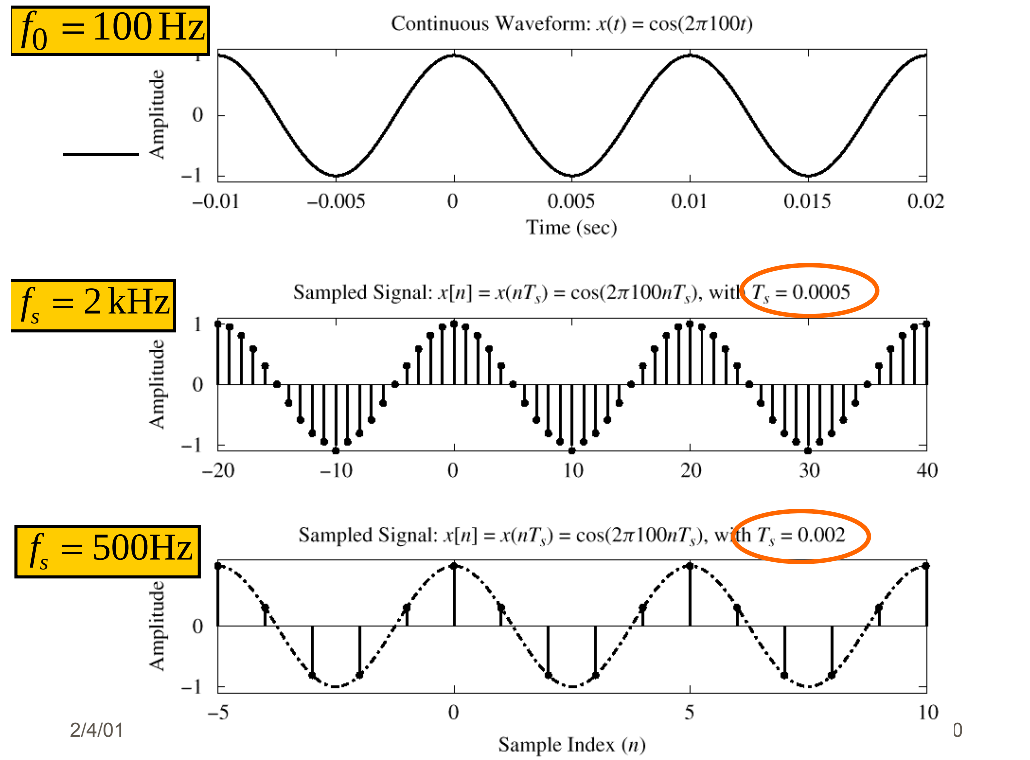
- SAMPLING PROCESS
  - Convert $x(t)$ to numbers $x[n]$
  - "n" is an integer; $x[n]$ is a sequence
  - "n" is the storage address in memory
- UNIFORM SAMPLING at $t = nT_s$
  - IDEAL: $x[n] = x(nT_s)$



SAMPLING RATE, f_s

- SAMPLING RATE (f_s)
  - $1/T_s$ = NUMBER of SAMPLES PER SECOND
  - T_s = 125 microsec --> f_s = 8000 samples/sec
    - UNITS ARE HERTZ: 8000 Hz
- UNIFORM SAMPLING at $t = nT_s = n/f_s$
  - IDEAL: $x[n] = x(nT_s) = x(n/f_s)$





Bandlimited Signals

- A bandlimited sum of sinusoids has the form

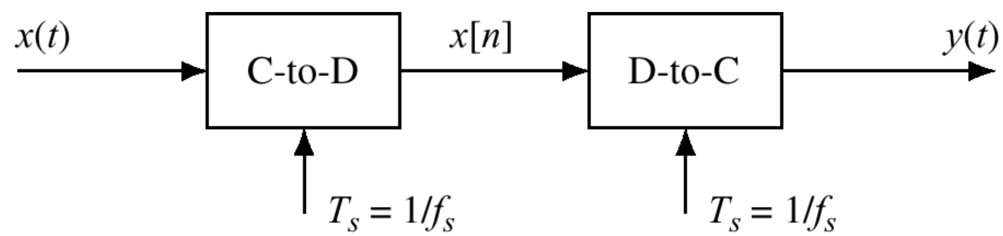
$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k) \text{ where } f_k \le f_{max}$$

- Our study of Fourier series has shown that:
  - Square waves are not bandlimited
  - Bandlimited signals are very smooth

SAMPLING THEOREM

Shannon Sampling Theorem

A continuous-time signal $x(t)$ with frequencies no higher than f_{max} can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{max}$.



$$y(t) = x(t) \quad \text{if } f_s > 2 f_{max}$$

STORING DIGITAL SOUND

- $x[n]$ is a SAMPLED SINUSOID
  - A list of numbers stored in memory
- CD rate is 44,100 samples per second
  - 16-bit samples
  - Stereo uses 2 channels
- Number of bytes for 1 minute is
  - 2 X (16/8) X 60 X 44100 = 10.584 Mbytes

DISCRETE-TIME SINUSOID

- Change x(t) into x[n] **DERIVATION**

$$x(t) = A\cos(\omega_0 t + \varphi)$$

$$x[n] = x(nT_s) = A\cos(\omega_0 nT_s + \varphi)$$

$$x[n] = A\cos((\omega_0 T_s)n + \varphi)$$

$$x[n] = A\cos(\hat{\omega}_0 n + \varphi)$$

$$\hat{\omega}_0 = \omega_0 T_s$$

DEFINE DIGITAL FREQUENCY

DIGITAL FREQUENCY $\hat{\omega}$

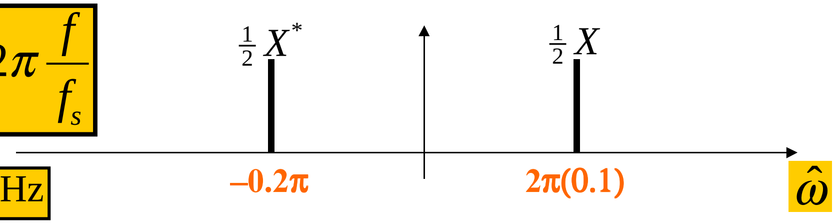
- $\hat{\omega}$ VARIES from **0** to **2π**, as f varies from 0 to the sampling frequency
- DIGITAL FREQUENCY is NORMALIZED
- UNITS are radians, **not** rad/sec

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}$$

SPECTRUM (DIGITAL)

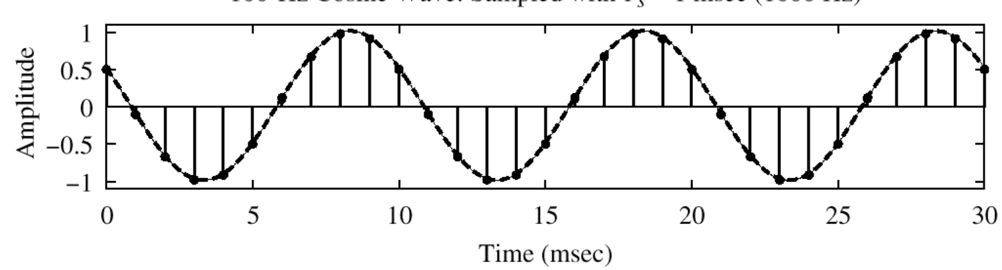
$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$f_s = 1\text{kHz}$



$$x[n] = \cos(2\pi(100)(n/1000) + \varphi) = \cos(0.2\pi n + \varphi)$$

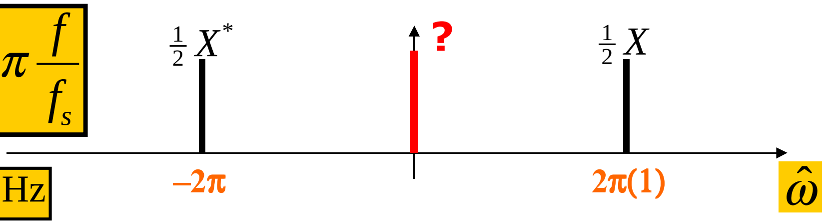
100-Hz Cosine Wave: Sampled with T_s = 1 msec (1000 Hz)



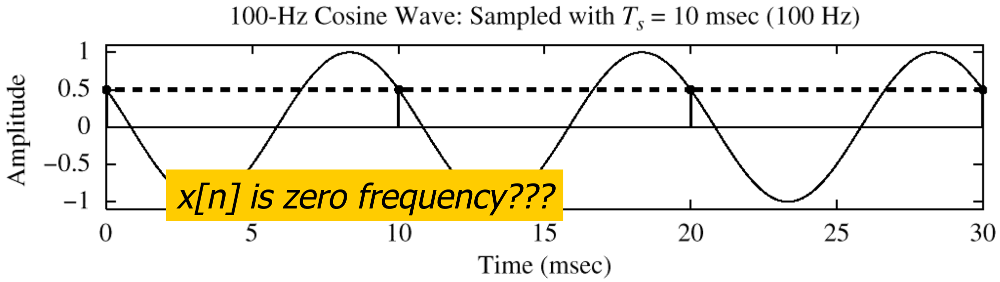
SPECTRUM (DIGITAL) ???

$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$$f_s = 100\,\text{Hz}$$



$$x[n] = \cos(2\pi(100)(n/100) + \varphi) \;= \cos(2\pi n + \varphi) = \cos(\varphi)$$



The REST of the STORY

- Spectrum of x[n] has more than one line for each complex exponential
  - Called **ALIASING**
  - **MANY SPECTRAL LINES**
- SPECTRUM is PERIODIC with period = **2π**
  - Because

$$A\cos(\hat{\omega} n + \varphi) = A\cos((\hat{\omega} + 2\pi)n + \varphi)$$

ALIASING DERIVATION

- Other Frequencies give the same $\hat{\omega}$

$$\text{If} \;\; x(t) = A\cos(2\pi(f + \ell f_s)t + \varphi)$$

$$\text{and we substitute:} \;\; t \leftarrow \tfrac{n}{f_s}$$

$$\text{then:} \;\; x[n] = A\cos(2\pi(f + \ell f_s)\tfrac{n}{f_s} + \varphi)$$

$$\text{or,} \;\; x[n] = A\cos(2\pi \tfrac{f}{f_s} n + 2\pi \ell n + \varphi)$$

ALIASING DERIVATION–2

- Other Frequencies give the same $\hat{\omega}$

$$\text{If} \;\; x(t) = A\cos(2\pi(f + \ell f_s)t + \varphi)$$

$$t \leftarrow \frac{n}{f_s}$$

$$\text{and we want:} \;\; x[n] = A\cos(\hat{\omega} n + \varphi)$$

$$\text{then:} \quad \hat{\omega} = \frac{2\pi(f + \ell f_s)}{f_s} = \frac{2\pi f}{f_s} + \frac{2\pi \ell f_s}{f_s}$$

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} + 2\pi\ell$$

ALIASING CONCLUSIONS

- ADDING f_s or $2f_s$ or $-f_s$ TO THE FREQ of x(t) gives exactly the same x[n]
- The samples, $x[n] = x(n/f_s)$ are EXACTLY THE SAME VALUES
- GIVEN x[n], WE CAN'T DISTINGUISH f_o FROM $(f_o + f_s)$ or $(f_o + 2f_s)$

NORMALIZED FREQUENCY

- DIGITAL FREQUENCY

Normalized Radian Frequency

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s} \boxed{+2\pi\ell}$$

Normalized Cyclic Frequency

$$\hat{f} = \hat{\omega}/(2\pi) = f T_s = f/f_s$$

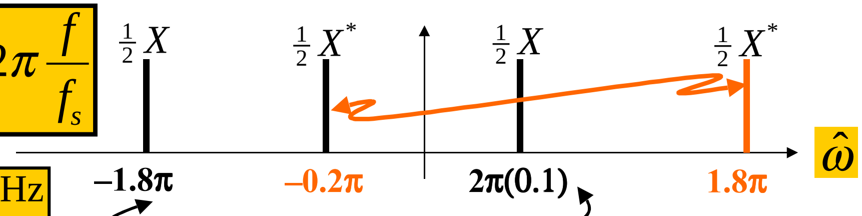
SPECTRUM for x[n]

- PLOT versus NORMALIZED FREQUENCY
- INCLUDE ALL SPECTRUM LINES
  - ALIASES
    - ADD MULTIPLES of 2π
    - SUBTRACT MULTIPLES of 2π
  - FOLDED ALIASES
    - (to be discussed later)
    - ALIASES of NEGATIVE FREQS

SPECTRUM (MORE LINES)

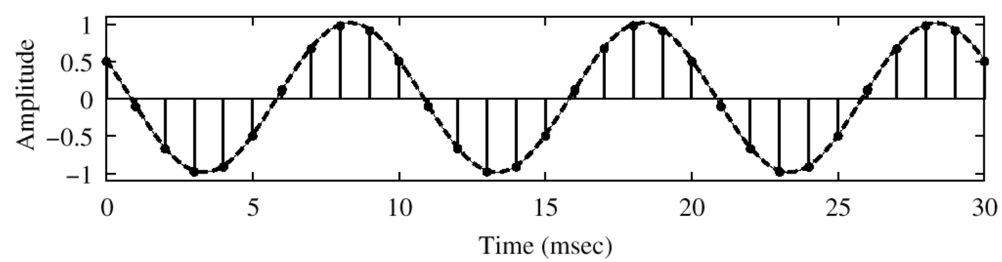
$$\hat{\omega} = 2\pi \frac{f}{f_s}$$

$f_s = 1\text{kHz}$



$$\omega = \hat{\omega}/T_s = \hat{\omega} f_s = 0.2\pi(1000) = 2\pi(100)$$

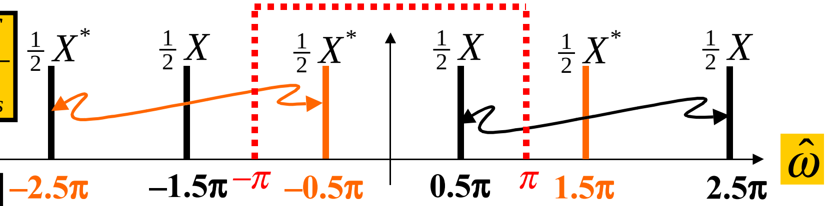
100-Hz Cosine Wave: Sampled with $T_s = 1$ msec (1000 Hz)



SPECTRUM (ALIASING CASE)

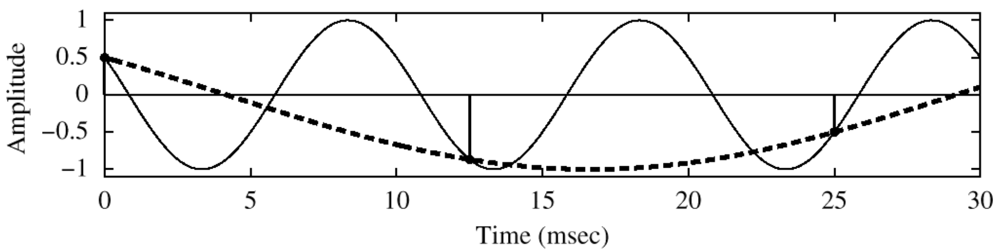
$\hat{\omega} = 2\pi \frac{f}{f_s}$

$f_s = 80\text{Hz}$



$\omega = \hat{\omega}/T_s = \hat{\omega} f_s = 0.5\pi(80) = 2\pi(20)$

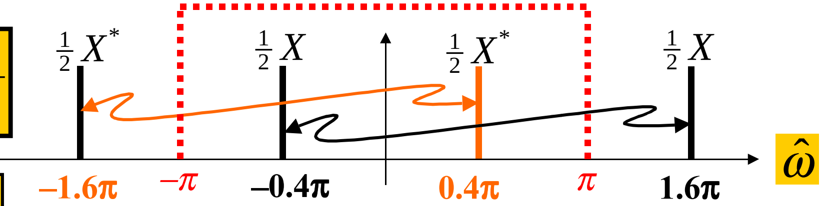
100-Hz Cosine Wave: Sampled with T_s = 12.5 msec (80 Hz)



SPECTRUM (FOLDING CASE)

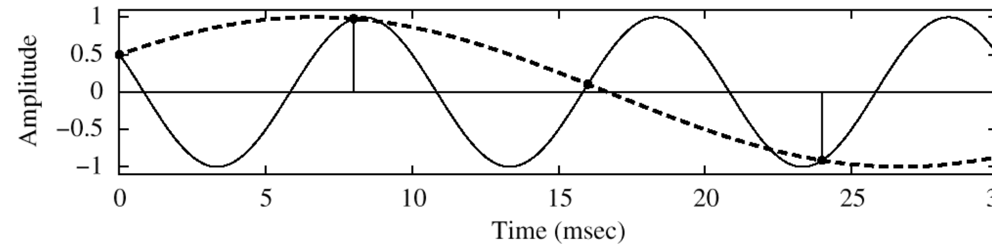
$\hat{\omega} = 2\pi \frac{f}{f_s}$

$f_s = 125\text{Hz}$



$\omega = \hat{\omega}/T_s = \hat{\omega} f_s = 0.4\pi(125) = 2\pi(25)$

100-Hz Cosine Wave: Sampled with T_s = 8 msec (125 Hz)



SAMPLING DEMO (Chap. 4)