EE-2025 Spring 2001

Lecture 16

Convolution (Continuous-Time)

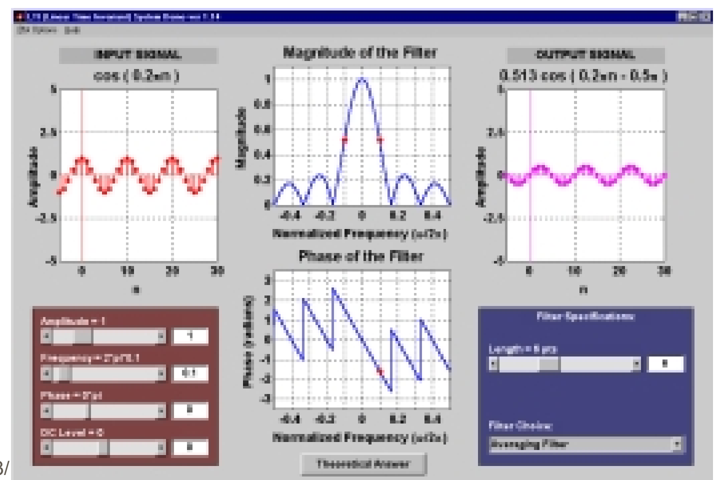
19-March-01

Info: Web-CT, Lab, HW

- Calendar:
  - Quiz #3 is 6-April
- Get NEW CHAPTERS
  - PDF or Bookstore
- Prob Set #8 and Lab #7 due this week
- Lab #8 is due next week
- Lab #9 this week in lab
- Lab QUIZ next WEEK

Lab #9 GUIs

- Download from WebCT: “MATLAB GUIs”



LECTURE

READING ASSIGNMENTS

- This Lecture:
  - Chapter 10, pp. 1020-1041
- Other Reading:
  - Recitation: Ch. 10, pp. 1020-1029
  - Next Lecture: Start reading Chapter 11

LECTURE OBJECTIVES

- Review of C-T LTI systems
- Evaluating convolutions
  - Examples
  - Impulses
- LTI Systems
  - Cascade and parallel connections
  - Stability and causality

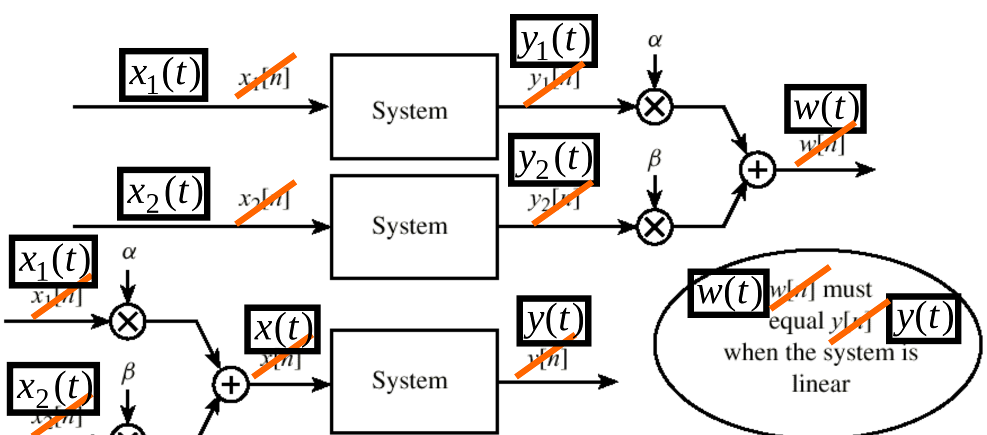
Linear and Time-Invariant (LTI) Systems

- If a continuous-time system is both linear and time-invariant, then the output *y(t)* is related to the input *x(t)* by a ***convolution integral***

$$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau = x(t)*h(t)$$

system.

Testing for Linearity



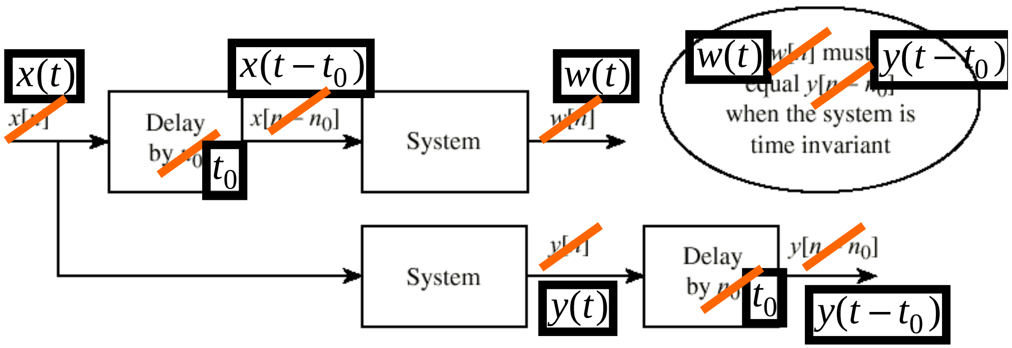
Convolution is Linear

- Substitute $x(t)=ax_1(t)+bx_2(t)$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty}[ax_1(\tau)+bx_2(\tau)]h(t-\tau)d\tau \\ &= a\int_{-\infty}^{\infty}x_1(\tau)h(t-\tau)d\tau + b\int_{-\infty}^{\infty}x_2(\tau)h(t-\tau)d\tau \\ &= ay_1(t)+by_2(t) \end{aligned}$$

Therefore convolution is linear.

Testing Time-Invariance



Convolution is Time-Invariant

- Substitute $x(t-t_0)$

$$w(t) = \int_{-\infty}^{\infty} h(\tau)x((t-\tau)-t_o)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)x((t-t_o)-\tau)d\tau$$

$$= y(t-t_o)$$

Ideal Delay: $y(t) = x(t-t_d)$

- Linear

$$ax_1(t-t_d)+bx_2(t-t_d) = ay_1(t)+by_2(t)$$

- and Time-Invariant

$$w(t) = x((t-t_0)-t_d)$$

$$y(t-t_0) = x((t-t_0)-t_d)$$

Ideal Delay: $y(t) = x(t-t_d)$

- Impulse response

$$h(t) = \delta(t-t_d)$$

- Convolution with an impulse

$$x(t) * \delta(t-t_d) = x(t-t_d)$$

Integrator: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

- Linear

$$\int_{-\infty}^{t} [a x_1(\tau) + b x_2(\tau)]d\tau = a y_1(t) + b y_2(t)$$

- And Time-Invariant

$$w(t) = \int_{-\infty}^{t} x(\tau - t_0)d\tau \qquad \text{let } \sigma = \tau - t_0$$

$$\Rightarrow \quad w(t) = \int_{-\infty}^{t-t_0} x(\sigma)d\sigma = y(t-t_0)$$

Integrator: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$

- Impulse response:

$$h(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

$$h(t) = u(t)$$

- Convolution with a step function:

$$\Rightarrow \quad x(t) * u(t) = \int_{-\infty}^{t} x(\sigma)d\sigma$$

Linear and Time-Invariant (LTI) Systems

- If a continuous-time system is both linear and time-invariant, then the output *y(t)* is related to the input *x(t)* by a ***convolution integral***

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

system.

Convolution of Impulses, etc.

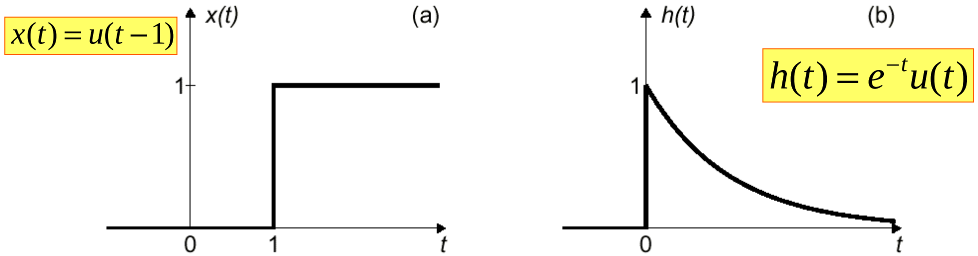
- Convolution of two impulses

$$\delta(t-t_1) * \delta(t-t_2) = \delta(t-t_1-t_2)$$

- Convolution of step and shifted impulse

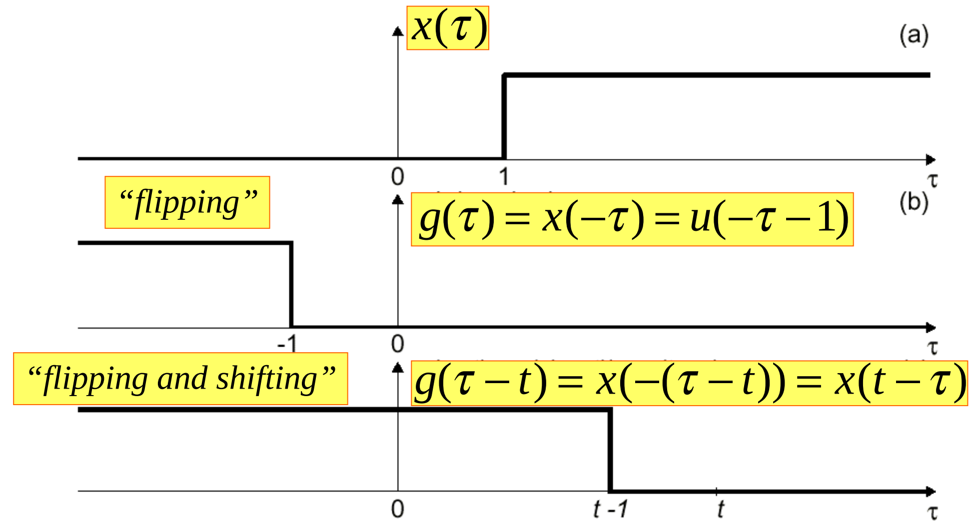
$$u(t) * \delta(t-t_0) = u(t-t_0)$$

Evaluating a Convolution

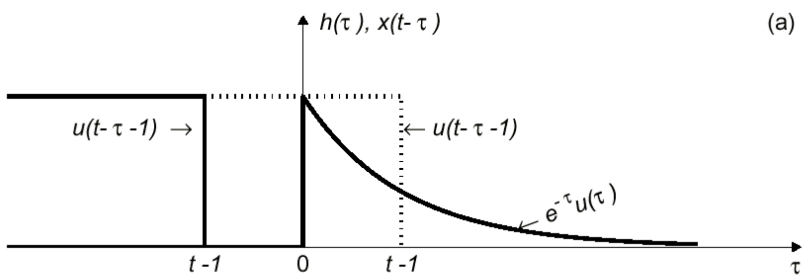


$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = h(t) * x(t)$$

"Flipping and Shifting"



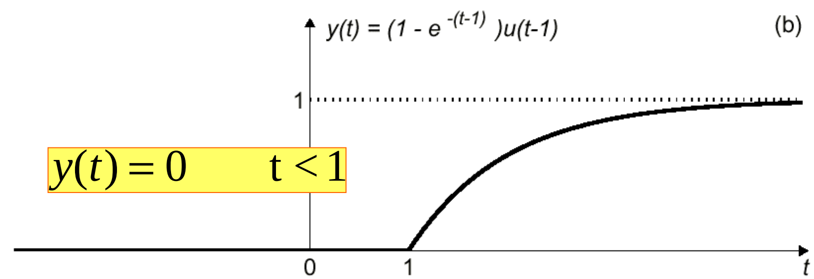
Evaluating the Integral



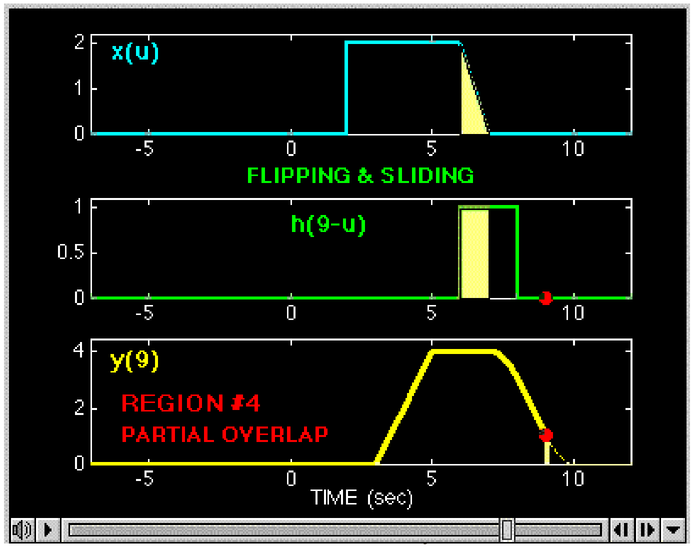
$$
\begin{aligned}
y(t) &= 0 & t-1<0 \\
&= \int_0^{t-1} e^{-\tau}d\tau & t-1 \geq 0
\end{aligned}
$$

Solution

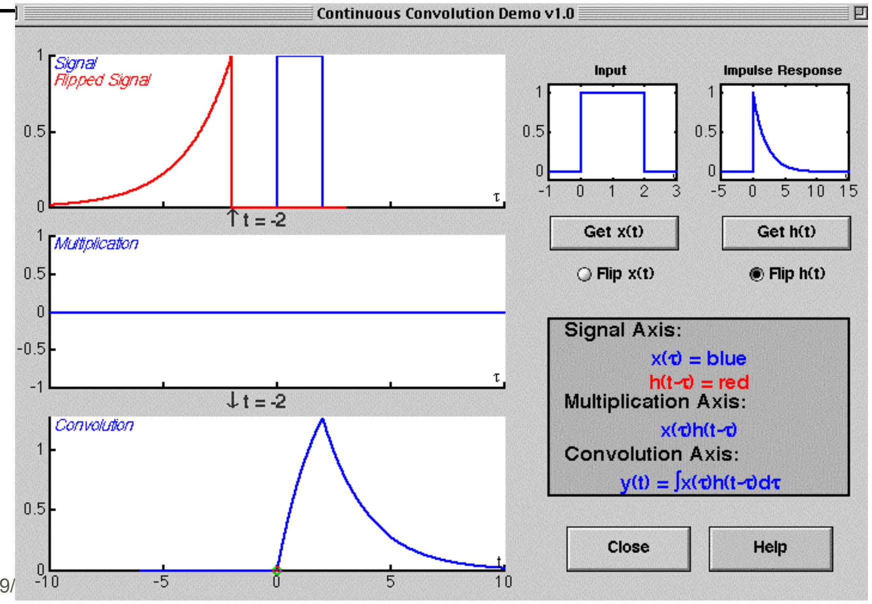
$$
\begin{aligned}
y(t) &= \int_0^{t-1} e^{-\tau}d\tau = -e^{-\tau}\Big|_0^{t-1} \\
&= 1 - e^{-(t-1)} \qquad t \geq 1
\end{aligned}
$$



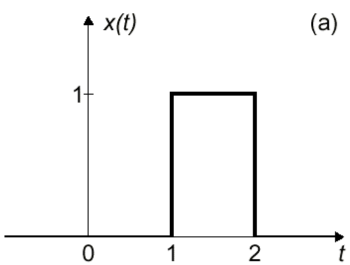
Convolution Demo

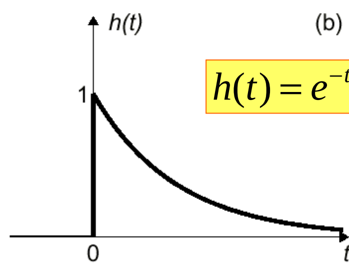


Convolution GUI



Another Convolution Example

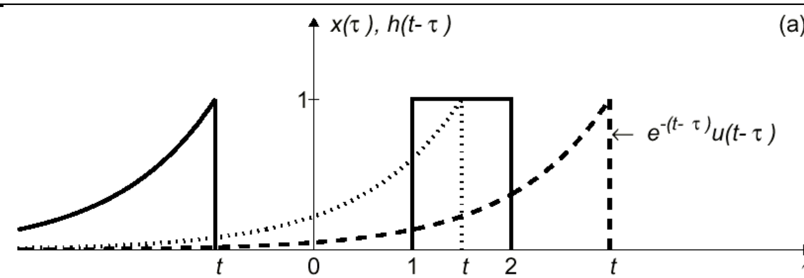




$$h(t) = e^{-t}u(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

Evaluating the Integral

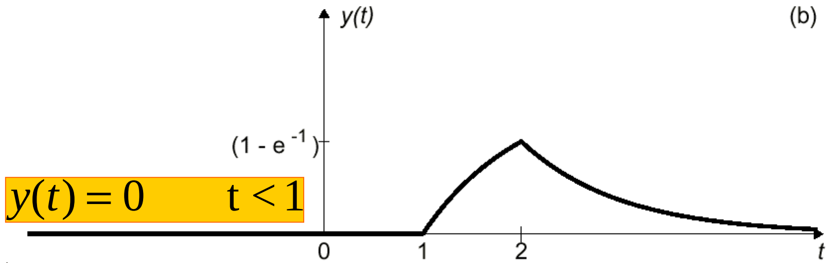


$$
\begin{aligned}
y(t) &= 0 & t < 1 \\
&= \int_{1}^{t} e^{-(t-\tau)}d\tau & 1 \le t \le 2 \\
&= \int_{1}^{2} e^{-(t-\tau)}d\tau & 2 \le t
\end{aligned}
$$

Solution

$$y(t) = \int_1^t e^{-(t-\tau)}d\tau = e^{-(t-\tau)}\Big|_1^t = 1 - e^{-(t-1)} \quad 1 \le t \le 2$$

$$= \int_1^2 e^{-(t-\tau)}d\tau = e^{-(t-\tau)}\Big|_1^2 = e^{-(t-2)} - e^{-(t-1)} \quad 2 \le t$$



Convolution is Commutative

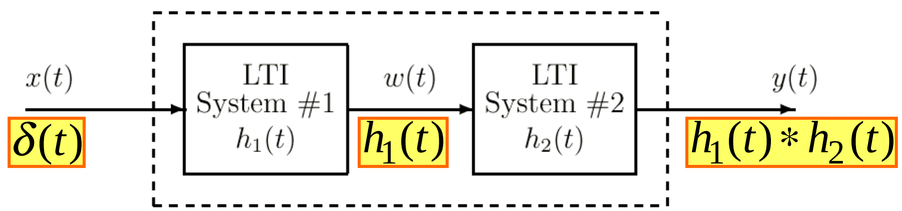
$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

let $\sigma = t - \tau$ and $d\sigma = -d\tau$

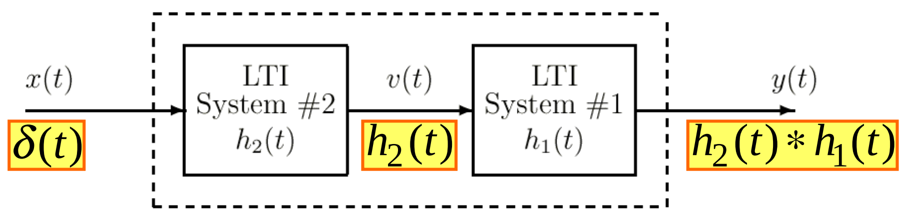
$$h(t) * x(t) = -\int_{\infty}^{-\infty} h(t-\sigma)x(\sigma)d\sigma$$

$$= \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\sigma = x(t) * h(t)$$

Cascade of LTI Systems

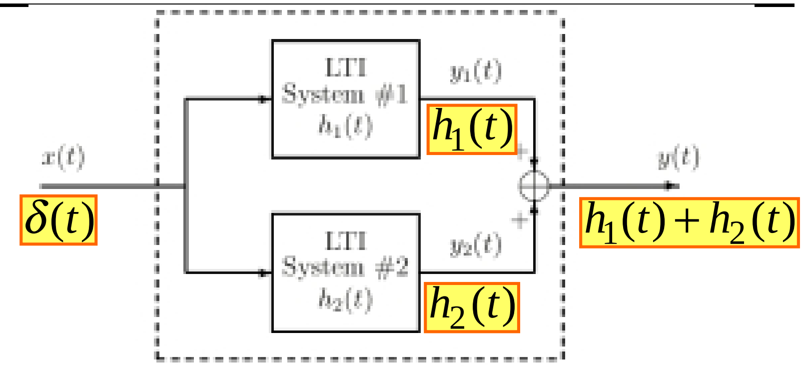


$$h(t) = h_1(t) * h_2(t) = h_2(t) * h_1(t)$$

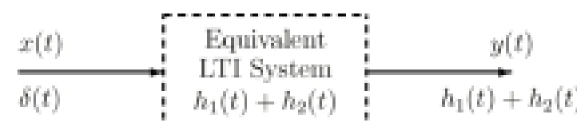


(b)

Parallel LTI Systems



$$h(t) = h_1(t) + h_2(t)(t$$



(b)

Stability

- A system is stable if every bounded input produces a bounded output.
- A continuous-time *LTI system* is stable if and only if

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

Causal Systems

- A system is causal if and only if $y(t_0)$ depends only on $x(\tau)$ for $\tau \leq t_0$.
- An LTI system is causal if and only if

$$h(t) = 0 \quad \text{for} \quad t < 0$$