EE-2025 Spring 2001

Lecture 18

Introduction to the Fourier Transform

26-March-01

Info: Web-CT, Lab, HW

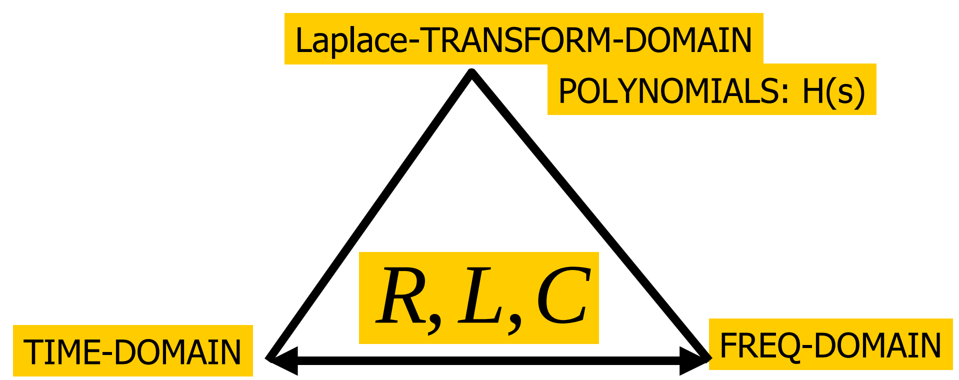
- Calendar:
  - Quiz #3 is 6-April
  - One page hand-written notes
  - Calculator
- Prob Set #9 is due this week
- Prob Set #10 and Lab #10 this week
- **Lab QUIZ this week**

Info: Lab #10

- Lab #10 is NUMERICAL FOURIER SERIES
- New MATLAB Functions:
- `fplot.m` and `quad8.m`
- `quad8(inline('t.*t'),0,1)`

Evaluate $\int_0^1 t^2 dt$ to get $\frac{1}{3}$

THREE DOMAINS: ANALOG



$y(t) = h(t) * x(t)$ ⟷ $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

LECTURE

READING ASSIGNMENTS

- This Lecture:
  - Chapter 12, pp. 1200-1214
- Other Reading:
  - Recitation: Chapter 11
    - And Chapter 12, pp. 1207–1214, 1218-1222, 1232–1234
  - Next Lecture: Chapter 12, pp. 1214-1218, 1223–1229, and 1236–1241

LECTURE OBJECTIVES

- Review
  - Frequency Response
  - Fourier Series
- Definition of Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

  - Relation
- Examples of Fourier transform pairs

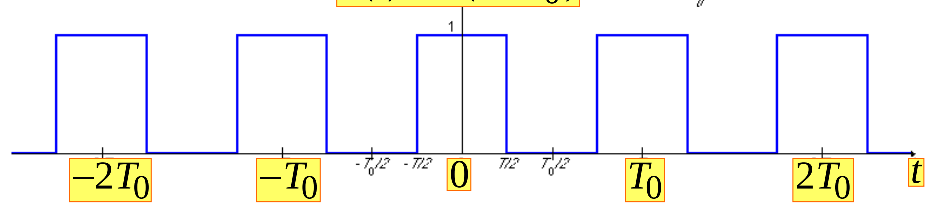
Everything = Sum of Sinusoids

- Square Pulse = Sum of Sinusoids
  - ???????????
- Finite Length
- Not Periodic

- Limit of Square Wave as Period –>infinity
  - Intuitive Argument

Fourier Series: Periodic *x(t)*

$$x(t) = x(t+T_0)$$



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

Fourier Synthesis

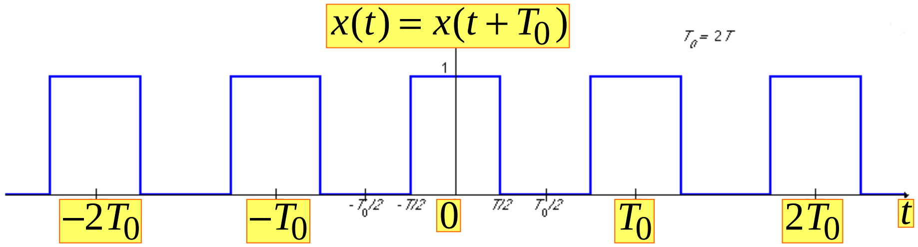
Fundamental Freq.

$$\omega_0 = 2\pi / T_0 = 2\pi f_0$$

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j\omega_0 k t}\,dt$$

Fourier Analysis

Square Wave Signal

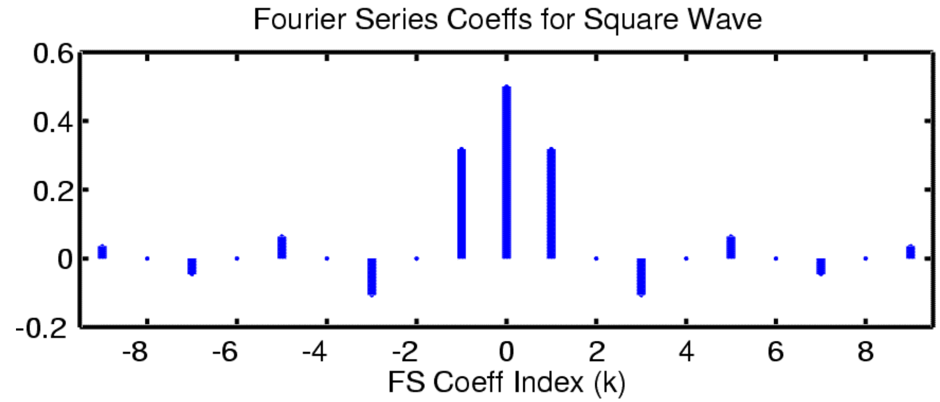


$$a_k = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} (1) e^{-j\omega_0 k t} dt$$

$$a_k = \left. \frac{e^{-j\omega_0 k t}}{-j\omega_0 k T_0} \right|_{-T_0/4}^{T_0/4} = \frac{e^{-j\pi k/2} - e^{j\pi k/2}}{-j2\pi k} = \frac{\sin(\pi k/2)}{\pi k}$$

Spectrum from Fourier Series

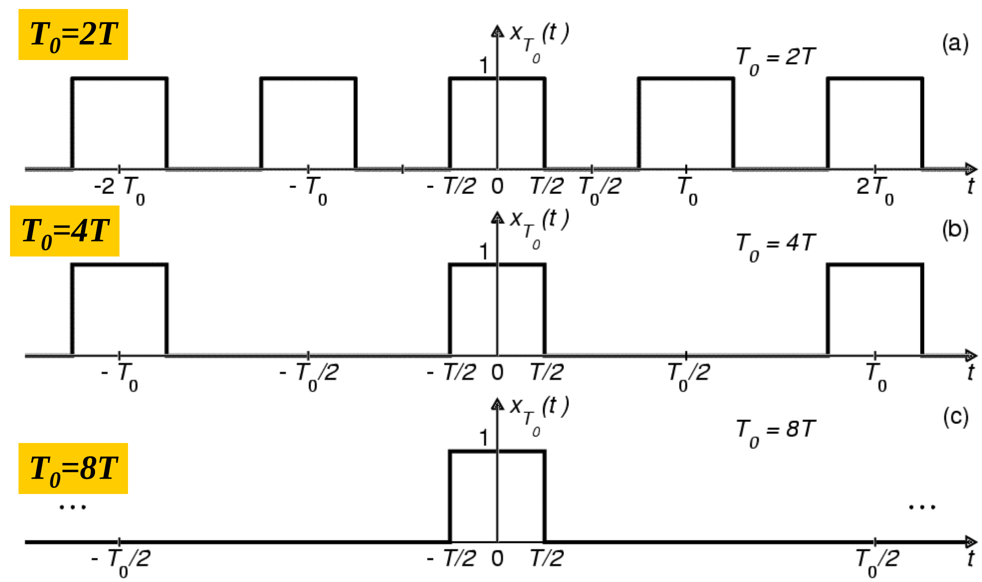
$$a_k = \frac{\sin(\pi k/2)}{\pi k} = \begin{cases} \neq 0 & k = 0, \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \end{cases}$$



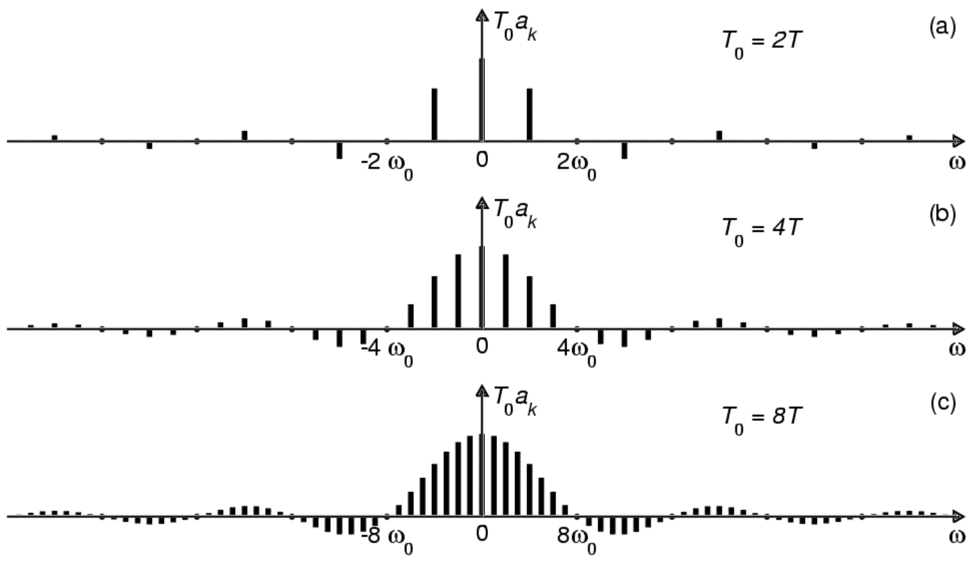
What if *x(t)* is not periodic?

- Sum of Sinusoids?
  - Non-harmonically related sinusoids
  - Would not be periodic, but would be non-zero for all *t*.
- Fourier transform
  - gives a "sum" (actually an **integral**) that involves **ALL** frequencies
  - can represent signals that are identically zero for negative *t*. !!!!!!!!!

Limiting Behavior of FS



Limiting Behavior of Spectrum



FS in the LIMIT (long period)

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} (T_0 a_k) e^{j\omega_0 k t} \left(\frac{2\pi}{T_0}\right) \mapsto x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

$$\lim_{T_0 \to \infty} \frac{2\pi}{T_0} = d\omega \qquad \lim_{T_0 \to \infty} \frac{2\pi}{T_0} k = \omega$$

$$T_0 a_k = \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_0 k t} dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

Fourier Transform

- For non-periodic signals

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

Example 1: $x(t) = e^{-at} u(t)$

$$X(j\omega) = \int_{0}^{\infty} e^{-at} e^{-j\omega t} dt = \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

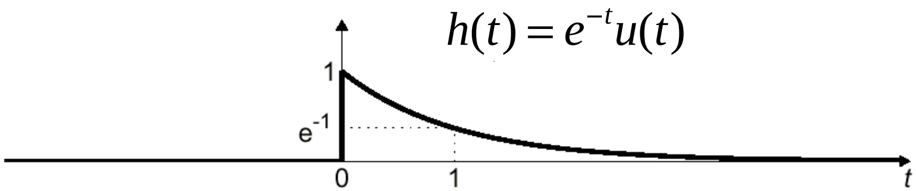
$$X(j\omega) = -\frac{e^{-at} e^{-j\omega t}}{a + j\omega}\Bigg|_0^{\infty} = \frac{1}{a + j\omega}$$

$a > 0$

$$X(j\omega) = \frac{1}{a + j\omega}$$

Frequency Response

- Fourier Transform of h(t) is the Frequency Response

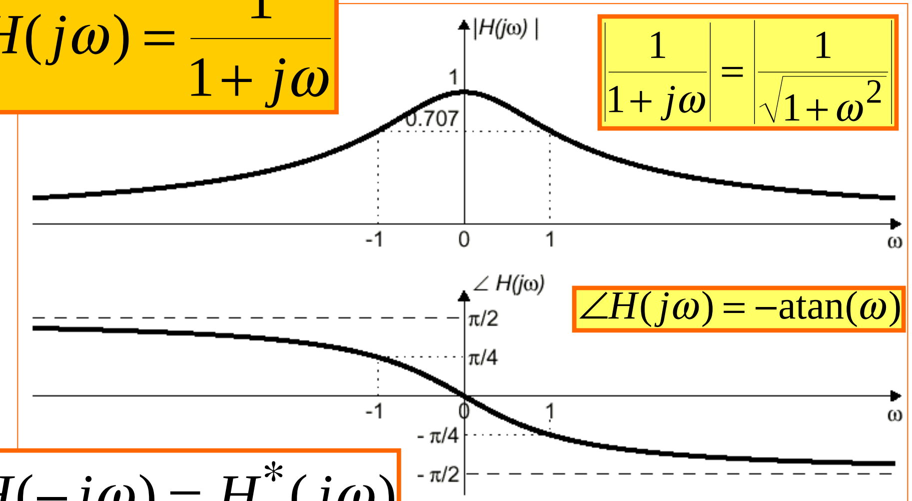


$$h(t)=e^{-t}u(t) \Leftrightarrow H(j\omega)=\frac{1}{1+j\omega}$$

Magnitude and Phase Plots

$$H(j\omega)=\frac{1}{1+j\omega}$$

$$\left|\frac{1}{1+j\omega}\right|=\frac{1}{\sqrt{1+\omega^2}}$$



$$\angle H(j\omega)=-\text{atan}(\omega)$$

$$H(-j\omega)=H^*(j\omega)$$

Example 2:

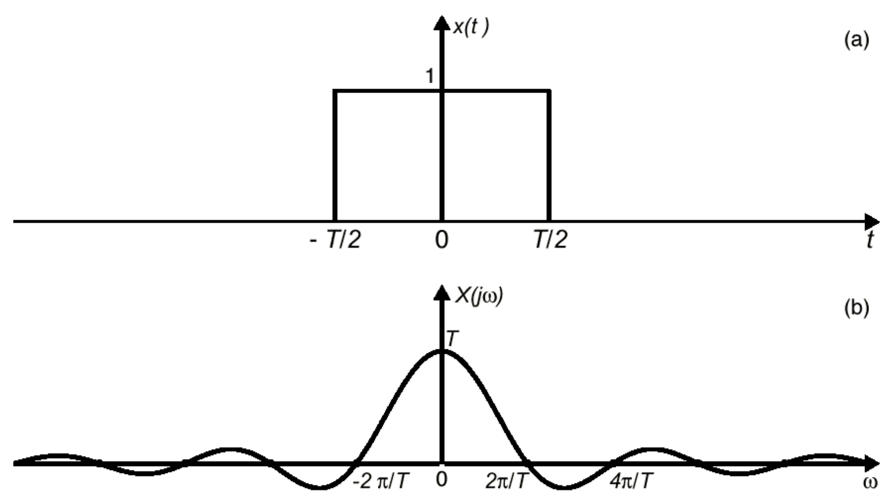
$$x(t)=\begin{cases}1 & |t|<T/2 \\ 0 & |t|>T/2\end{cases}$$

$$X(j\omega)=\int_{-T/2}^{T/2}(1)e^{-j\omega t}dt=\int_{-T/2}^{T/2}e^{-j\omega t}dt$$

$$X(j\omega)=\left.\frac{e^{-j\omega t}}{-j\omega}\right|_{-T/2}^{T/2}=\frac{e^{-j\omega T/2}-e^{j\omega T/2}}{-j\omega}$$

$$X(j\omega)=\frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t)=\begin{cases}1 & |t|<T/2 \\ 0 & |t|>T/2\end{cases} \Leftrightarrow X(j\omega)=\frac{\sin(\omega T/2)}{(\omega/2)}$$



Example 3:

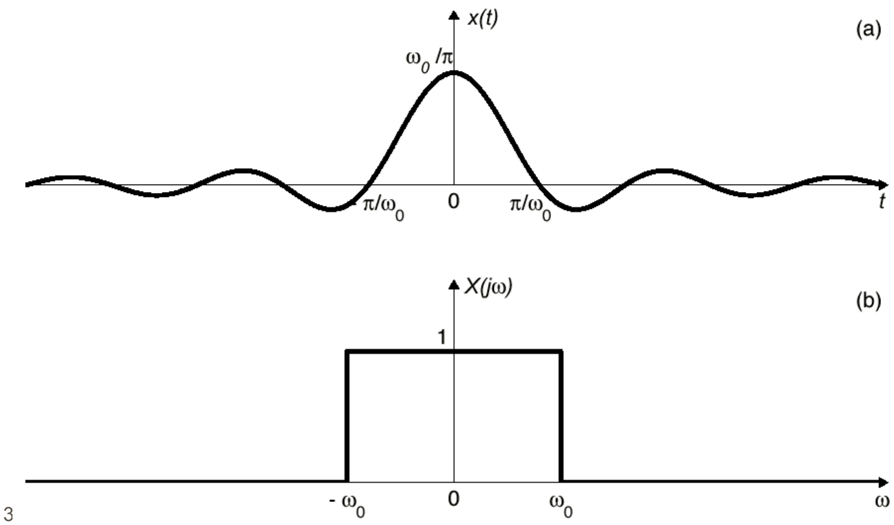
$$X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega t} d\omega$$

$$x(t) = \frac{1}{2\pi} \left. \frac{e^{j\omega t}}{jt} \right|_{-\omega_0}^{\omega_0} = \frac{1}{2\pi} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{jt}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$



Example 4:

$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Shifting Property of the Impulse

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0}$$

$$x(t) = \delta(t) \Leftrightarrow X(j\omega) = 1$$

Example 5: $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)$$

$$x(t) = \cos(\omega_0 t) \Leftrightarrow X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$x(t) = \cos(\omega_0 t) \Leftrightarrow X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

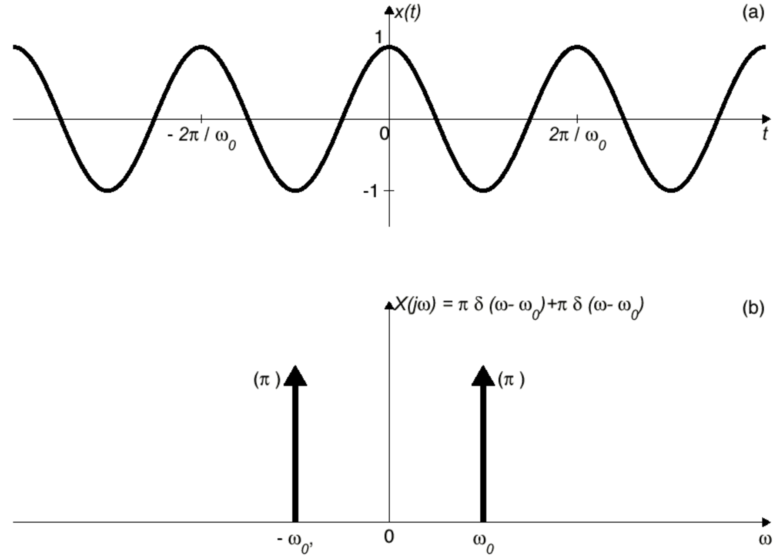


Table of Fourier Transforms

$$x(t) = e^{-at}u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$