

Lecture 19

Fourier Transform Properties

30-March-01

Info: Web-CT, Lab, HW

- Calendar:
 - Quiz #3 is 6-April (next Friday)
 - One page hand-written notes Calculator
 - Covers Prob Sets #8, 9, 10
- **REVIEW: Thursday (5-Apr) at 8PM**
- Prob Set #10 is due NEXT WEEK
 - Solution will be posted Thursday

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Info: Labs

- Labs
 - #10: Numerical Fourier Series due next week
 - #11: Fourier-Based Design, work on next two weeks, due week of 16-April
 - #12: Communication/AM, assigned week of 16-April, due week of 23-April

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Strategy for using the FT

- Develop a set of known Fourier transform pairs.
- Develop a set of “theorems” or properties of the Fourier transform.
- Develop skill in formulating the problem in either the time-domain or the frequency-domain, *which ever leads to the simplest solution.*

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LECTURE

READING ASSIGNMENTS

■ This Lecture:

- Chapter 12, pp. 1214-1218, 1223–1229, and 1236–1241

■ Other Reading:

- Recitation: All of Chapter 12
- Next Lecture: Chapter 13, pp. 1317-1331

LECTURE OBJECTIVES

■ The Fourier transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

- More examples of Fourier transform pairs
- Basic properties of Fourier transforms
 - Convolution property
 - Multiplication property

Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega$$

Fourier Synthesis
(Inverse Transform)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

Fourier Analysis
(Forward Transform)

Time - domain \Leftrightarrow Frequency - domain

$$x(t) \Leftrightarrow X(j\omega)$$

WHY use the Fourier transform?

- Manipulate the **“Frequency Spectrum”**
- Analog Communication Systems
 - AM: Amplitude Modulation; FM
 - What are the **“Building Blocks”** ?
 - **Abstract Layer**, not implementation
- Ideal Filters: mostly BPFs
- Frequency Shifters
 - Modulators, or Multipliers: $x(t)p(t)$

Table of Fourier Transforms

$$x(t) = e^{-at} u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T / 2)}{(\omega / 2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

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Example 3:

$$X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega t} d\omega$$

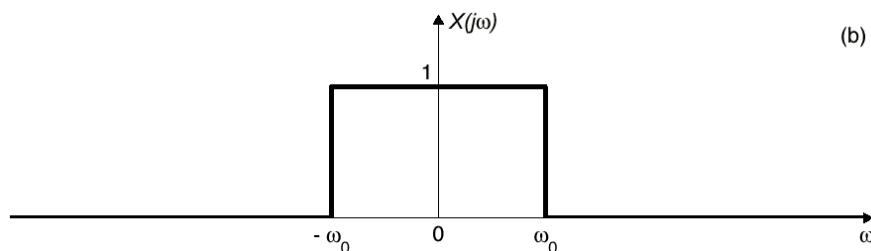
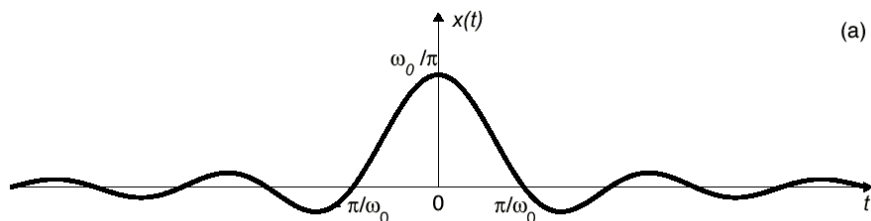
$$x(t) = \frac{1}{2\pi} \frac{e^{j\omega t}}{jt} \Big|_{-\omega_0}^{\omega_0} = \frac{1}{2\pi} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{jt}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)}$$

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$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$



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Example 5: $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)$$

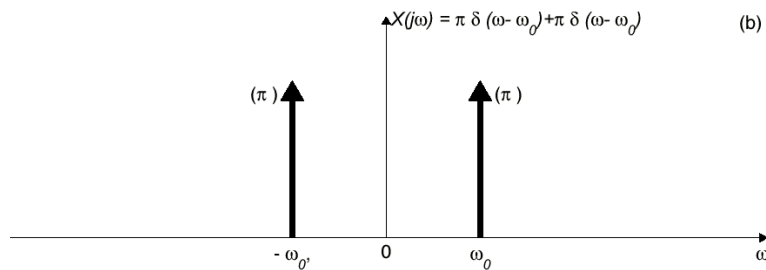
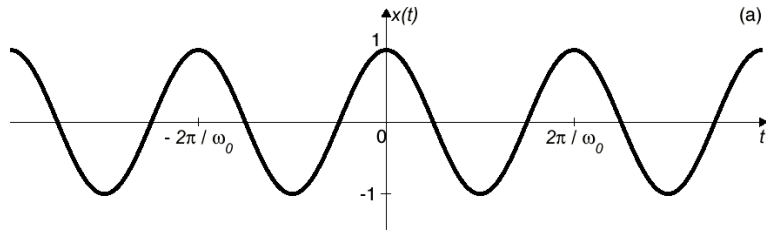
$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

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$$x(t) = \cos(\omega_0 t) \Leftrightarrow X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



Fourier Transform of a General Periodic Signal

If $x(t)$ is periodic with period T_0 ,

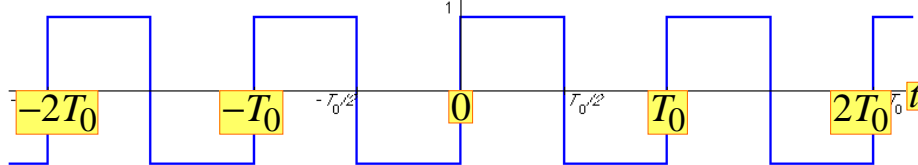
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

Therefore, since $e^{jk\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - k\omega_0)$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Square Wave Signal

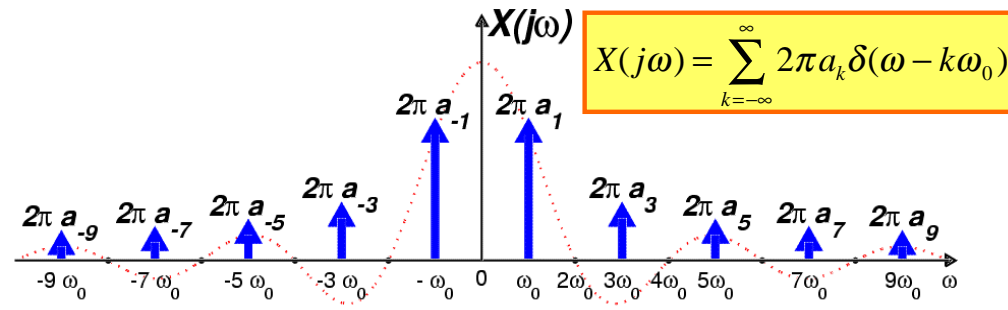
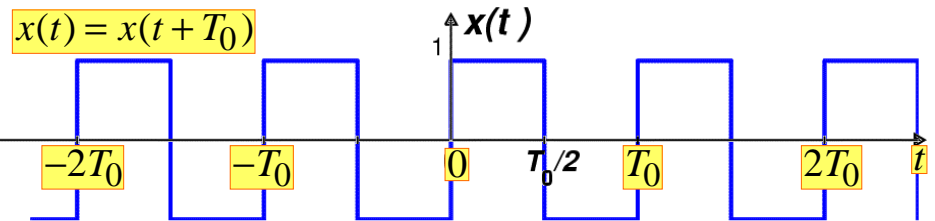
$$x(t) = x(t + T_0)$$



$$a_k = \frac{1}{T_0} \int_0^{T_0/2} (1) e^{-j\omega_0 k t} dt + \frac{1}{T_0} \int_{T_0/2}^{T_0} (-1) e^{-j\omega_0 k t} dt$$

$$a_k = \frac{e^{-j\omega_0 k t}}{-j\omega_0 k T_0} \Big|_0^{T_0/2} - \frac{e^{-j\omega_0 k t}}{-j\omega_0 k T_0} \Big|_{T_0/2}^{T_0} = \frac{1 - e^{-j\pi k}}{j\pi k}$$

Square Wave Fourier Transform



$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

FT of Impulse Train

- The periodic impulse train is

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\omega_0 = 2\pi / T_0$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-j\omega_0 t} dt = \frac{1}{T_0} \quad \text{for all } k$$

$$\therefore P(j\omega) = \left(\frac{2\pi}{T_0} \right) \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

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Table of Easy FT Properties

Linearity Property

$$ax_1(t) + bx_2(t) \Leftrightarrow aX_1(j\omega) + bX_2(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Frequency Shifting

$$x(t) e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j\left(\frac{\omega}{a}\right))$$

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Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

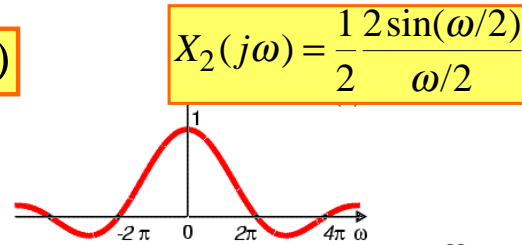
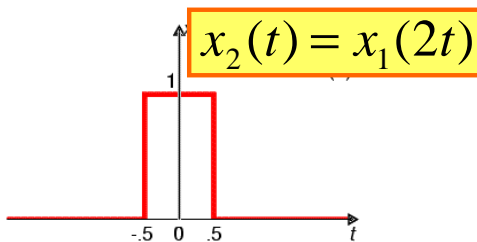
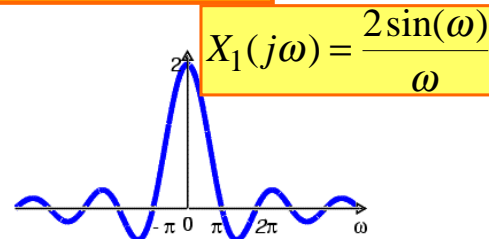
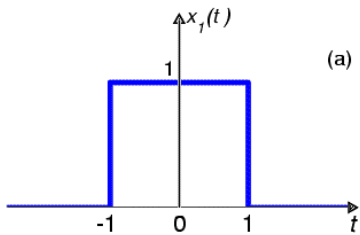
$$\begin{aligned} \int_{-\infty}^{\infty} x(t - t_d) e^{-j\omega t} dt &= \int_{-\infty}^{\infty} x(\tau) e^{-j\omega(\tau + t_d)} d\tau \\ &= e^{-j\omega t_d} X(j\omega) \end{aligned}$$

For example, $e^{-a(t-5)} u(t-5) \Leftrightarrow \frac{e^{-j\omega 5}}{a + j\omega}$

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Scaling Property

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\frac{\omega}{a}))$$



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Uncertainty Principle

- Try to make $x(t)$ shorter
 - Then $X(j\omega)$ will get wider
 - Narrow pulses have wide bandwidth
- Try to make $X(j\omega)$ narrower
 - Then $x(t)$ will have longer duration
- **Cannot simultaneously reduce time duration and bandwidth**

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Significant FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Differentiation Property

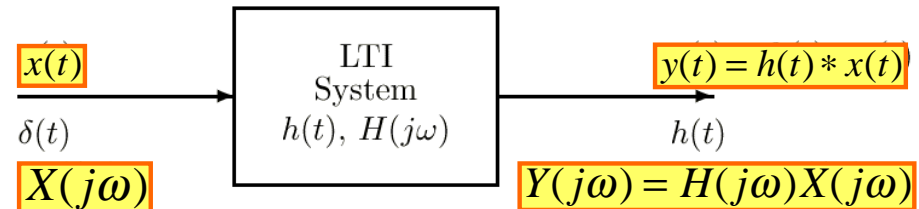
$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

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Convolution Property



- Convolution in the time-domain

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau$$

corresponds to **MULTIPLICATION** in the frequency-domain $Y(j\omega) = H(j\omega)X(j\omega)$

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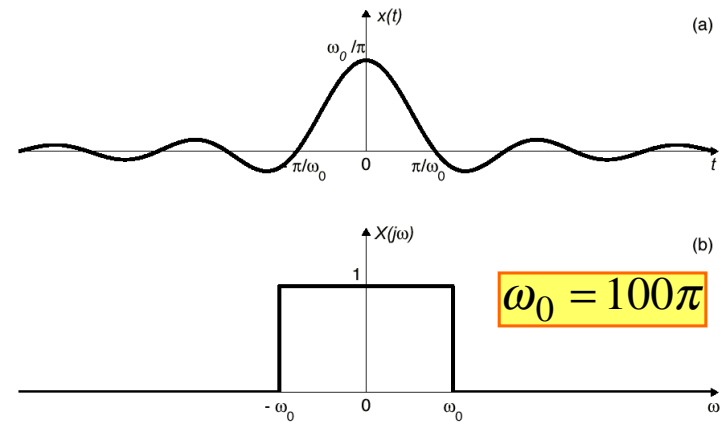
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Convolution Example

- Bandlimited Input Signal
 - "sinc" function
- Ideal LPF (Lowpass Filter)
 - $h(t)$ is a "sinc"
- Output is Bandlimited
 - Convolve "sincs"

Ideally Bandlimited Signal

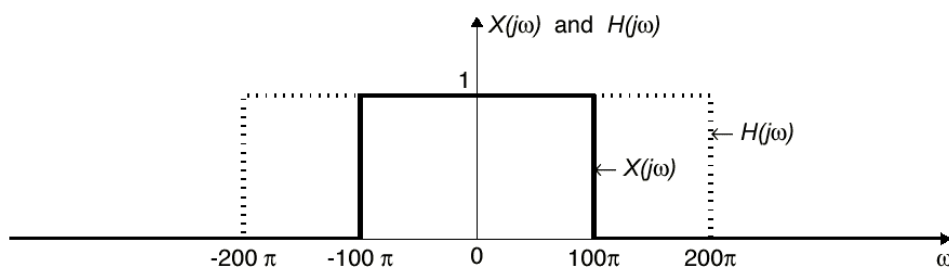
$$x(t) = \frac{\sin(100\pi t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < 100\pi \\ 0 & |\omega| > 100\pi \end{cases}$$



Convolution Example

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$\frac{\sin(100\pi t)}{\pi t} * \frac{\sin(200\pi t)}{\pi t} = \frac{\sin(100\pi t)}{\pi t}$$

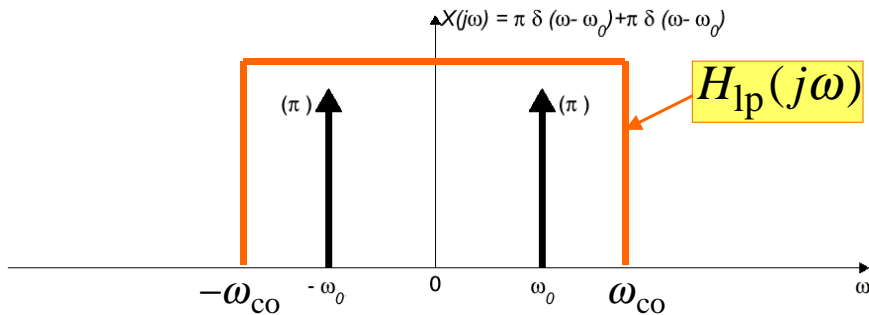


Cosine Input to LTI System

$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= H(j\omega)[\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)] \\ &= H(j\omega_0)\pi\delta(\omega - \omega_0) + H(-j\omega_0)\pi\delta(\omega + \omega_0) \end{aligned}$$

$$\begin{aligned} y(t) &= H(j\omega_0)\frac{1}{2}e^{j\omega_0 t} + H(-j\omega_0)\frac{1}{2}e^{-j\omega_0 t} \\ &= H(j\omega_0)\frac{1}{2}e^{j\omega_0 t} + H^*(j\omega_0)\frac{1}{2}e^{-j\omega_0 t} \\ &= |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0)) \end{aligned}$$

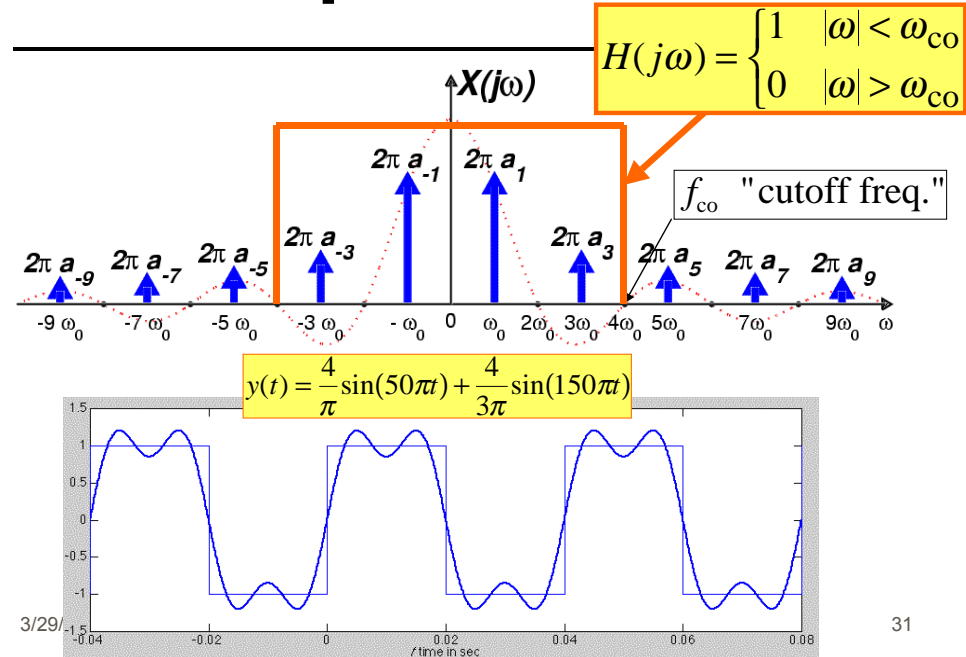
Ideal Lowpass Filter



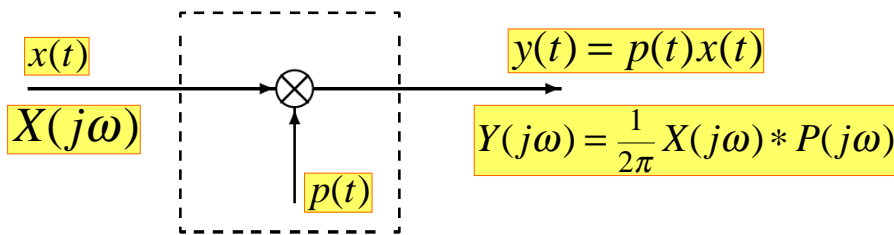
$$y(t) = x(t) \quad \text{if } \omega_0 < \omega_{co}$$

$$y(t) = 0 \quad \text{if } \omega_0 > \omega_{co}$$

Ideal Lowpass Filter



Signal Multiplier (Modulator)



- Multiplication in the time-domain corresponds to convolution in the frequency-domain.

$$Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta)P(j(\omega - \theta))d\theta$$

Frequency Shifting Property

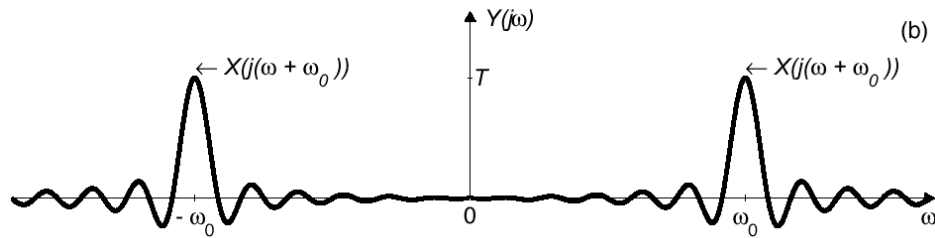
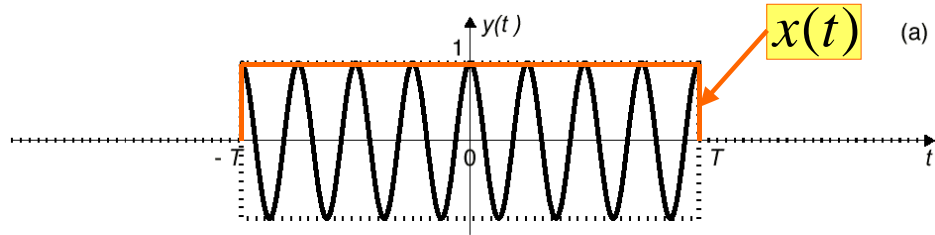
$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

$$\int_{-\infty}^{\infty} e^{-j\omega_0 t} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \omega_0)t} dt = X(j(\omega - \omega_0))$$

$$y(t) = \frac{\sin 7t}{\pi} e^{j\omega_0 t} \Leftrightarrow Y(j\omega) = \begin{cases} 1 & \omega_0 - 7 < \omega < \omega_0 + 7 \\ 0 & \text{elsewhere} \end{cases}$$

$$y(t) = x(t) \cos(\omega_0 t) \Leftrightarrow$$

$$Y(j\omega) = \frac{1}{2} X(j(\omega - \omega_0)) + \frac{1}{2} X(j(\omega + \omega_0))$$



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Differentiation Property

$$\begin{aligned} \frac{dx(t)}{dt} &= \frac{d}{dt} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \right) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega) X(j\omega) e^{j\omega t} d\omega \end{aligned}$$

$$\begin{aligned} \frac{d}{dt} (e^{-at} u(t)) &= -ae^{-at} u(t) + e^{-at} \delta(t) \\ &= \delta(t) - ae^{-at} u(t) \end{aligned}$$

$$\Leftrightarrow \frac{j\omega}{a + j\omega}$$

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