EE-2025 Spring-2001

Lecture 22

Discrete-Time Filtering of Continuous-Time Signals

13-April-01

Info: Web-CT, Lab, HW

- Calendar: Final Exams
  - Period 15, Tuesday, 4-May
  - Arrange CONFLICTS soon.
- Reading Assignment:
  - You should have read Chapters 2-8 of DSP First and all chapters of the notes.
- Prob Set #13 - due the last day;
  - will be discussed in recitations and covered on FINAL EXAM

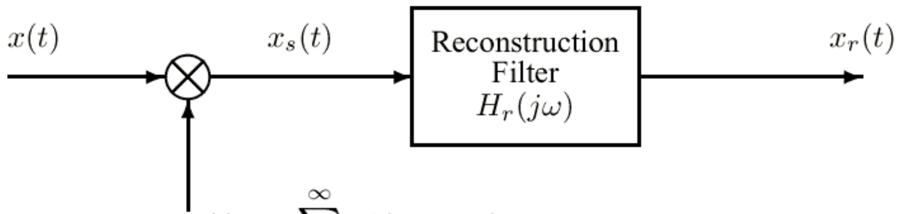
LAB QUIZ #3

- Lab Quiz Coverage:
  - Labs 7, 8, 9, 10, and 11
  - Especially the Warm-ups
- Course Evaluations during last week
  - THREE
    - Two for GT: Lecture & Recitation
    - ******** One on Web-CT ***********
  - AM Demodulation evaluations

LECTURE OBJECTIVES

- Discrete-Time Filtering of Continuous-Time Signals
  - Basic Configuration
    - CT Input -> A/D -> DT System -> D/A -> CT Output
- EFFECTIVE FREQUENCY RESPONSE
  - For Bandlimited Input Signals
  - Relies on the General Version of the Sampling Theorem

Impulse Train Sampling



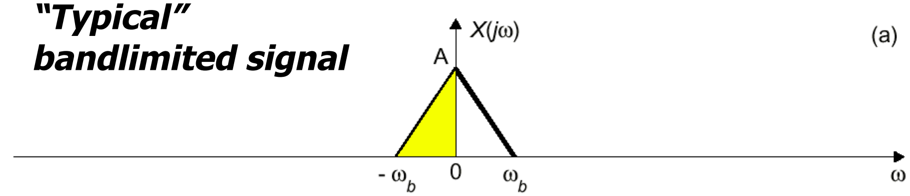
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s)$$

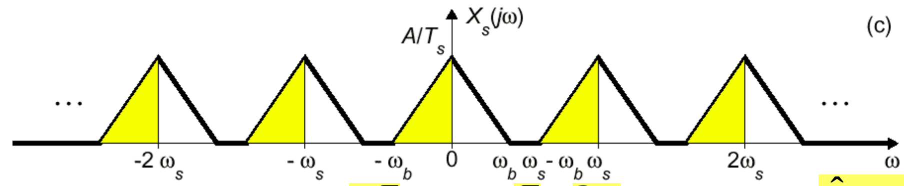
$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Frequency-Domain Representation of Sampling

"Typical" bandlimited signal

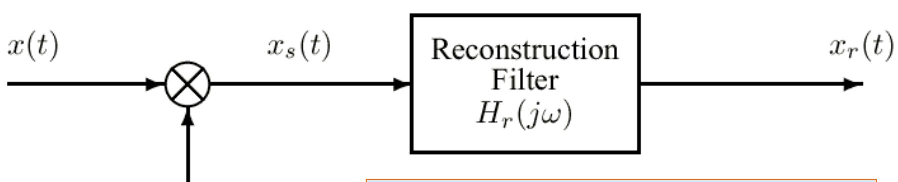


$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$



$-\pi$ π 2π $\hat{\omega} = \omega T_s$

Reconstruction of *x*(*t*)

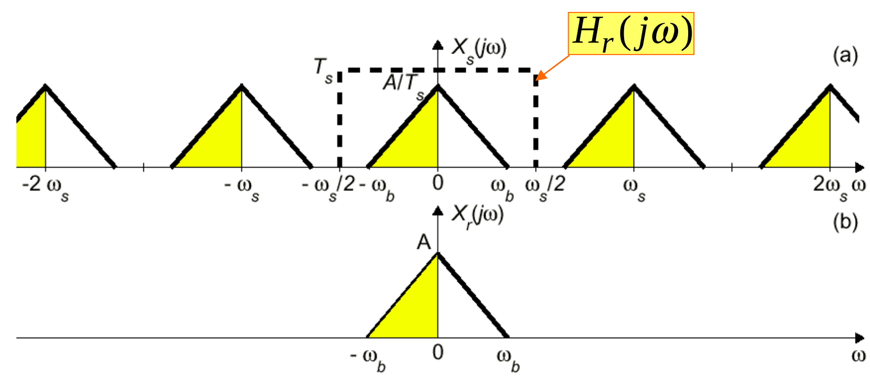


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \qquad x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega)X_s(j\omega)$$

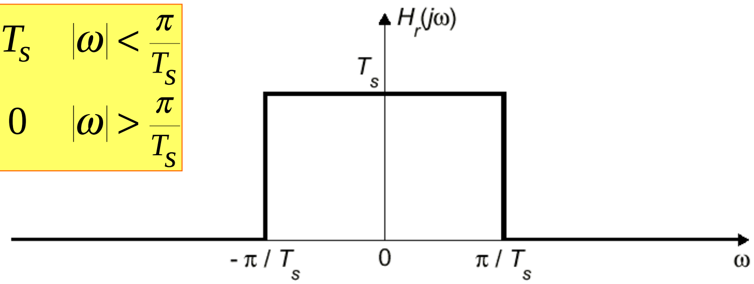
Reconstruction in the Frequency-Domain



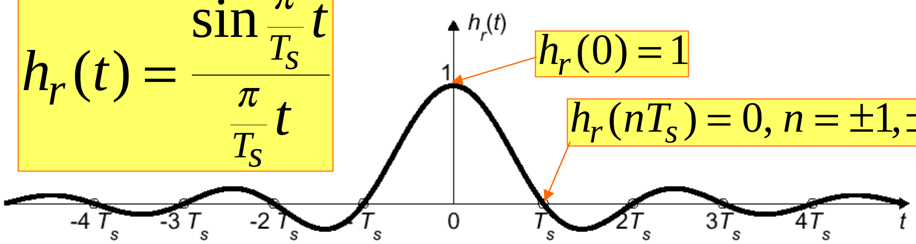
- If $\omega_s > 2\omega_b$, the copies of $X(j\omega)$ do not overlap, so $X_r(j\omega) = H_r(j\omega)\, X_s(j\omega)$.

Ideal Reconstruction Filter

$$H_r(j\omega) = \begin{cases} T_s & |\omega| < \frac{\pi}{T_s} \\ 0 & |\omega| > \frac{\pi}{T_s} \end{cases}$$



$$h_r(t) = \frac{\sin \frac{\pi}{T_s} t}{\frac{\pi}{T_s} t}$$



$$h_r(0) = 1$$

$$h_r(nT_s) = 0, n = \pm 1, \pm 2, \ldots$$

Signal Reconstruction

$$x_r(t) = h_r(t) * x_s(t) = h_r(t) * \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) h_r(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolation formula

Shannon Sampling Theorem

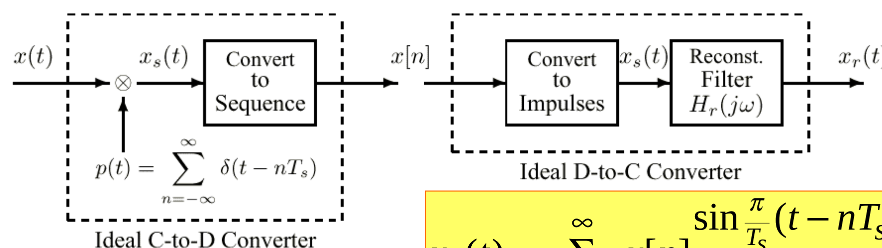
- PERFECT RECONSTRUCTION of BANDLIMITED SIGNALS

A signal $x(t)$ with bandlimited Fourier transform such that $X(j\omega) = 0$ for $|\omega| \geq \omega_b$ can be reconstructed exactly from samples taken with sampling rate $\omega_s = 2\pi/T_s \geq 2\omega_b$ using the following bandlimited interpolation formula:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolation formula

Ideal C-to-D and D-to-C



$$x[n] = x(nT_s)$$

Ideal Sampler

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolator

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega) X_s(j\omega)$$

General Frequency-Domain Analysis of C-to-D Conversion

$$x_s(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega-k\omega_s))$$

$$\omega_s = \frac{2\pi}{T_s}$$

$$X_s(j\omega) = \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\omega nT_s} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n}$$

$$x[n] = x(nT_s) \qquad \hat{\omega} = \omega T_s$$

Discrete-Time Fourier Transform and *z*-Transform

$$X_s(j\omega) = \sum_{n=-\infty}^{\infty} x(nT_s)e^{-j\omega nT_s} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega T_s n}$$

$$\hat{\omega} = \omega T_s$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

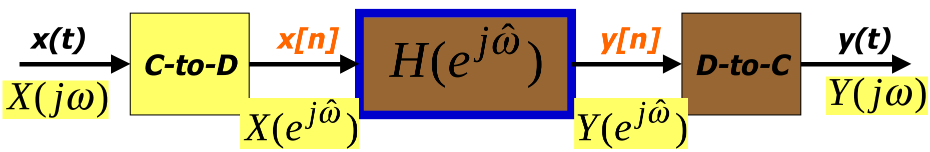
***z*-Transform**

$$X(z)\big|_{z=e^{j\omega T_s}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega T_s n} = X(e^{j\omega T_s})$$

DTFT

$$\Rightarrow X(e^{j\omega T_s}) = X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega-k\omega_s))$$

C-to-D Converter

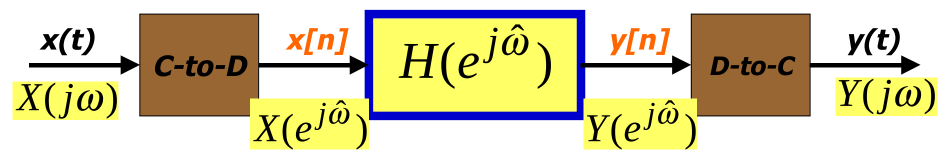


$$x[n] = x(nT_s)$$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega-k\omega_s))$$

$$X(e^{j\omega T_s}) = X(z)\big|_{z=e^{j\omega T_s}} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega T_s n} = X_s(j\omega)$$

LTI DT System



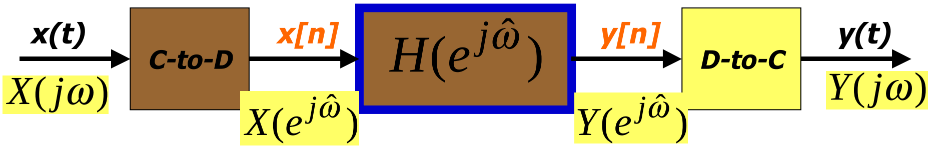
$$Y(z) = H(z)X(z)$$

$$Y(e^{j\hat{\omega}}) = H(e^{j\hat{\omega}})X(e^{j\hat{\omega}})$$

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

$$Y(e^{j\omega T_s}) = H(e^{j\omega T_s})X(e^{j\omega T_s})$$

D-to-C Converter

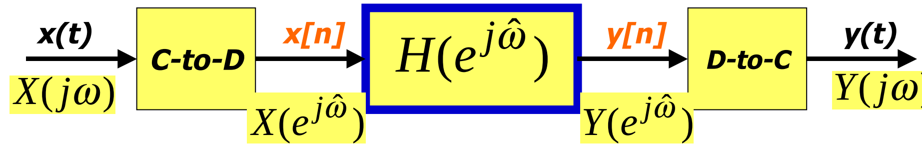


$$y(t) = \sum_{n=-\infty}^{\infty} y[n]h_r(t - nT_s)$$

$$Y(j\omega) = \sum_{n=-\infty}^{\infty} y[n]H_r(j\omega)e^{-j\omega T_s n} = H_r(j\omega)\sum_{n=-\infty}^{\infty} y[n]e^{-j\omega T_s n}$$

$$Y(j\omega) = H_r(j\omega)Y(e^{j\omega T_s})$$

Putting it All Together



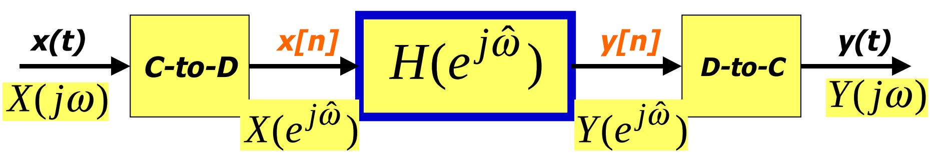
$$Y(j\omega) = H_r(j\omega)Y(e^{j\omega T_s}) = H_r(j\omega)H(e^{j\omega T_s})X(e^{j\omega T_s})$$

$$Y(j\omega) = H_r(j\omega)H(e^{j\omega T_s})\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

If no aliasing occurs in sampling x(t), then it follows that

$$Y(j\omega) = H(e^{j\omega T_s})X(j\omega) = H_{\text{eff}}(j\omega)X(j\omega)$$

DT Filtering of CT Signals

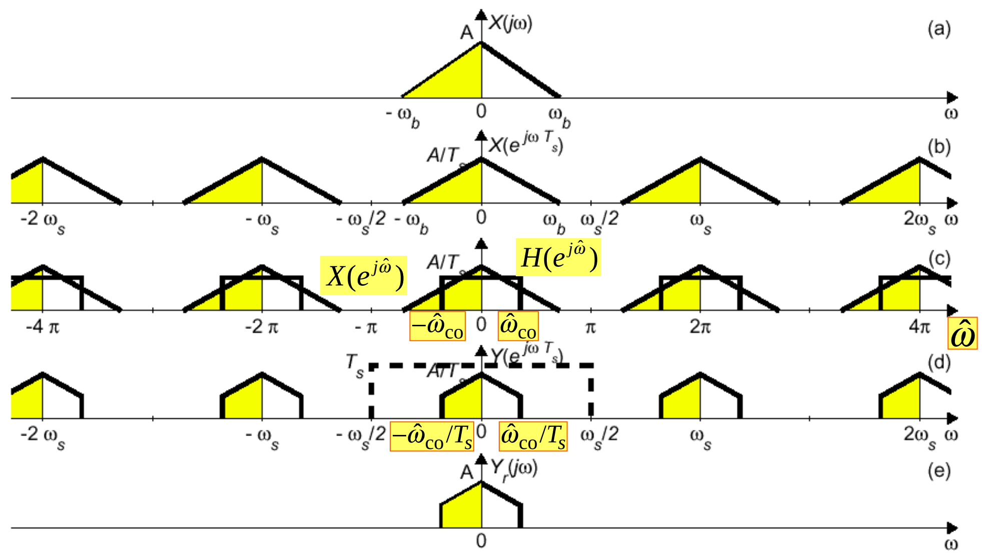


If no aliasing occurs in sampling x(t), then it follows that

$$Y(j\omega) = H_{\text{eff}}(j\omega)X(j\omega)$$

$$H_{\text{eff}}(j\omega) = \begin{cases} H(e^{j\omega T_s}) & |\omega| < \frac{1}{2}\omega_s \\ 0 & |\omega| > \frac{1}{2}\omega_s \end{cases}$$

Illustration of DT Filtering of a CT Signal



EFFECTIVE Freq. Response

- Assume NO Aliasing, then
  - ANALOG FREQ <--> DIGITAL FREQ

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

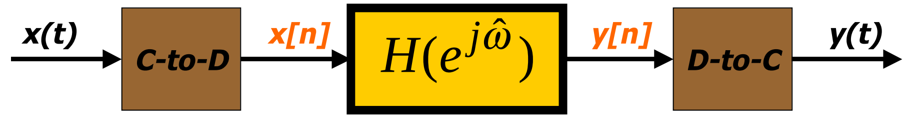
DIGITAL FILTER

- So, we can plot:
- Scaled Freq. Axis

$$H(e^{j\omega T_s}) \text{ vs. } \omega$$

ANALOG FREQUENCY

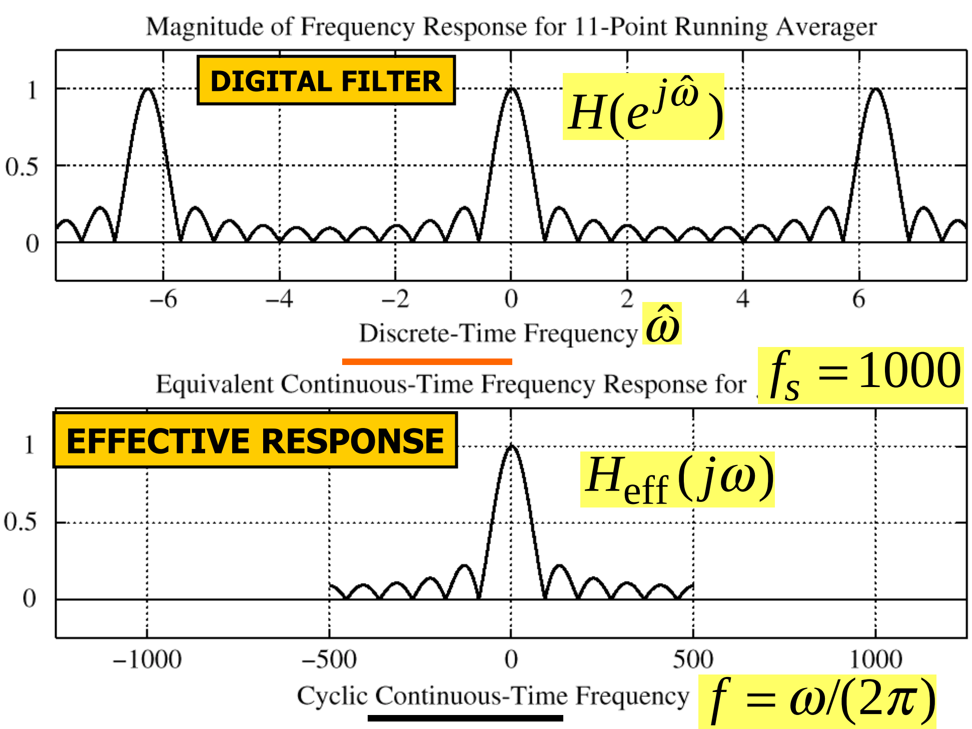
11-pt AVERAGER Example



$$y[n] = \frac{1}{11}\sum_{k=0}^{10} x[n-k]$$

$$H(e^{j\hat{\omega}}) = \frac{1}{11}\sum_{k=0}^{10} e^{-j\hat{\omega}k} = \frac{\sin(\hat{\omega}11/2)}{11\sin(\hat{\omega}/2)} e^{-j\hat{\omega}5}$$

$$H_{\text{eff}}(j\omega) = \frac{\sin(\omega T_s 11/2)}{11\sin(\omega T_s/2)} e^{-j\omega T_s 5} \qquad |\omega| < \frac{\pi}{T_s}$$



Example

- Input signal:

$$x(t) = \cos(500\pi t) \Leftrightarrow$$

$$X(j\omega) = \pi\delta(\omega - 500\pi) + \pi\delta(\omega + 500\pi)$$

- If f_s = 1000 Hz, no aliasing occurs

$$Y(j\omega) = H_{\text{eff}}(j\omega) X(j\omega)$$

$$Y(j\omega) = H_{\text{eff}}(j\omega)[\pi\delta(\omega - 500\pi) + \pi\delta(\omega + 500\pi)]$$

$$Y(j\omega) = H_{\text{eff}}(j500\pi)\pi\delta(\omega - 500\pi) + H_{\text{eff}}(-j500\pi)\pi\delta(\omega + 500\pi)$$

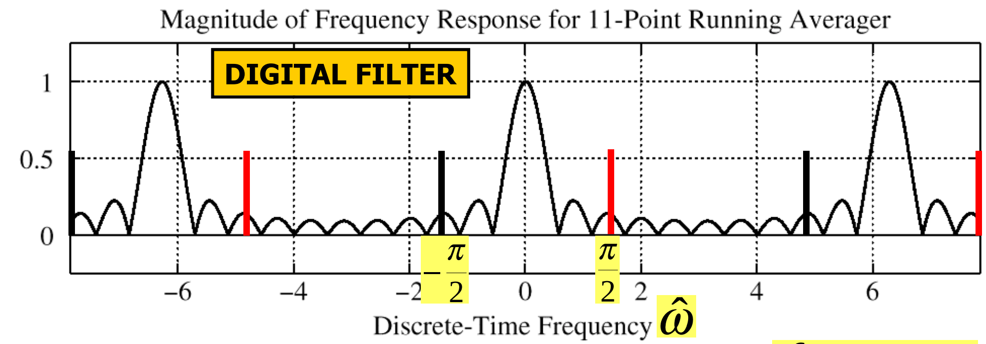
EVALUATE Freq. Response

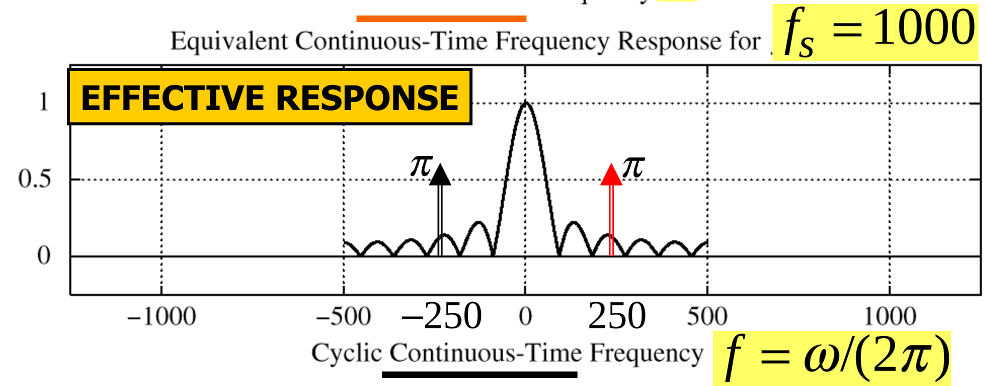
$$H(e^{j\hat{\omega}}) = \frac{\sin(\hat{\omega}11/2)}{11\sin(\hat{\omega}/2)}e^{-j\hat{\omega}5}$$

$$H_{\text{eff}}(j500\pi) = H(e^{j500\pi/1000}) = H(e^{j0.5\pi})$$

$$H(e^{j0.5\pi}) = \frac{\sin((0.5\pi)11/2)}{11\sin(0.5\pi/2)}e^{-j(0.5\pi)5}$$

$$= \frac{\sin(2.75\pi)}{11\sin(0.25\pi)}e^{-j2.5\pi} = 0.0909e^{-j0.5\pi}$$





Example (continued)

$$Y(j\omega) = H_{\text{eff}}(j\omega)[\pi\delta(\omega - 500\pi) + \pi\delta(\omega + 500\pi)]$$

$$Y(j\omega) = H_{\text{eff}}(j500\pi)\pi\delta(\omega - 500\pi) + H_{\text{eff}}(-j500\pi)\pi\delta(\omega + 500\pi)]$$

$$Y(j\omega) = 0.0909e^{-j.5\pi}\pi\delta(\omega - 500\pi) + 0.0909e^{j.5\pi}\pi\delta(\omega + 500\pi)]$$

$$y(t) = 0.1818\cos(500\pi t - 0.5\pi)$$