EE-2025 Spring 2001

Lecture 23

IIR Filters: Feedback & H(z)

16-April-01

Info: HW and Labs

- Reading Assignment:
  - Chapter 8 in DSP First
- Prob Set #12 - due this week
  - Prob Set #13 - due 27-April
- Lab #11 due this week of 16-April
  - Lab #12 due next week of 23-April
  - ALL Labs must be turned in by 27-April
- Office Hours this week 5-6pm, M & Th or after class today and Friday.

Final Exam Info

- Final Exam: Period 15, 4-May, 2:50pm
- **Report CONFLICTS immediately !!!!**
  - E.g., 3 exams in one day
  - Middle exam should be rescheduled
- Reviews will be held on Weds. And Thurs.
  - 7pm in ECE Auditorium

READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, pp. 249-263
- Other Reading:
  - Recitation: Ch. 8, pp. 261-272
    - POLES & ZEROS
  - Next Lecture: Chapter 8, pp. 269-282

LECTURE OBJECTIVES

INFINITE IMPULSE RESPONSE FILTERS

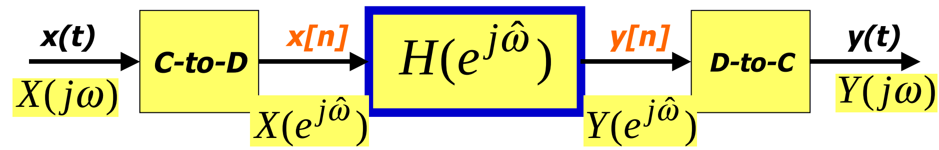
Define **IIR** DIGITAL Filters

Have **FEEDBACK**: use PREVIOUS OUTPUTS

$$y[n] = \sum_{\ell=1}^{N} a_\ell y[n-\ell] + \sum_{k=0}^{M} b_k x[n-k]$$

- Show how to compute the output y[n]
  - FIRST-ORDER CASE (N=1)
  - Z-transform: Impulse Response h[n] <--> H(z)

DT Filtering of CT Signals



If no aliasing occurs in sampling x(t), then it follows that

$$Y(j\omega) = H_{\text{eff}}(j\omega)X(j\omega)$$

$$H_{\text{eff}}(j\omega) = \begin{cases} H(e^{j\omega T_s}) & |\omega| < \frac{1}{2}\omega_s \\ 0 & |\omega| > \frac{1}{2}\omega_s \end{cases}$$

Example

- Assume BL input with no aliasing in sampling.
- Assume first-difference system
  $y[n] = x[n] - x[n-1]$

$$\Leftrightarrow H(z) = 1 - z^{-1} \Leftrightarrow H(e^{j\hat{\omega}}) = 1 - e^{-j\hat{\omega}}$$

- Effective frequency response

$$H_{\text{eff}}(j\omega) = \begin{cases} 1 - e^{-j\omega T_s} & |\omega| < \pi/T_s \\ 0 & |\omega| > \pi/T_s \end{cases}$$

Example (continued)

- Effective frequency response

$$H_{\text{eff}}(j\omega) = \begin{cases} 1 - e^{-j\omega T_s} & |\omega| < \pi/T_s \\ 0 & |\omega| > \pi/T_s \end{cases}$$

$$H_{\text{eff}}(j\omega) = \begin{cases} 2j\sin(\omega T_s / 2)e^{-j\omega T_s/2} & |\omega| < \pi/T_s \\ 0 & |\omega| > \pi/T_s \end{cases}$$

$$y(t) = x(t) - x(t - T_s) \quad \text{(if } x(t) \text{ is BL)}$$

Quick Review: L-pt Averager

$$y[n] = \sum_{k=0}^{L-1} \tfrac{1}{L} x[n-k]$$

IMPULSE RESPONSE

$$h[n] = \sum_{k=0}^{L-1} \tfrac{1}{L} \delta[n-k]$$

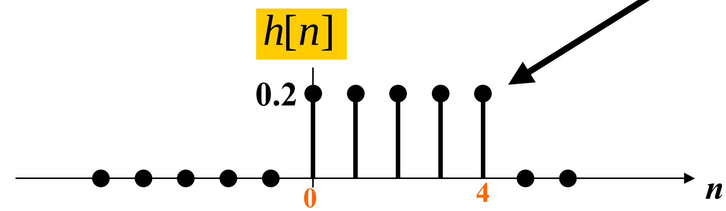
SYSTEM FUNCTION

$$H(z) = \sum_{n=0}^{L-1} \tfrac{1}{L} z^{-n}$$

MATH FORMULA for h[n]

- Use SHIFTED IMPULSES to write h[n]

$$h[n] = \tfrac{1}{5}\delta[n] + \tfrac{1}{5}\delta[n-1] + \tfrac{1}{5}\delta[n-2] + \tfrac{1}{5}\delta[n-3] + \tfrac{1}{5}\delta[n-4]$$



$$\{b_k\} = \{\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}\}$$

LTI: Convolution

- Output = Convolution of x[n] & h[n]
  - NOTATION: y[n] = x[n]*h[n]
  - Here is the FIR case:

FINITE LIMITS

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

Same as $\mathbf{b_k}$

FINITE LIMITS

L-pt RUNNING AVG H(z)

$$H(z) = \sum_{k=0}^{L-1} \tfrac{1}{L} z^{-k} = \frac{1-z^{-L}}{L(1-z^{-1})} = \frac{z^L - 1}{Lz^{L-1}(z-1)}$$

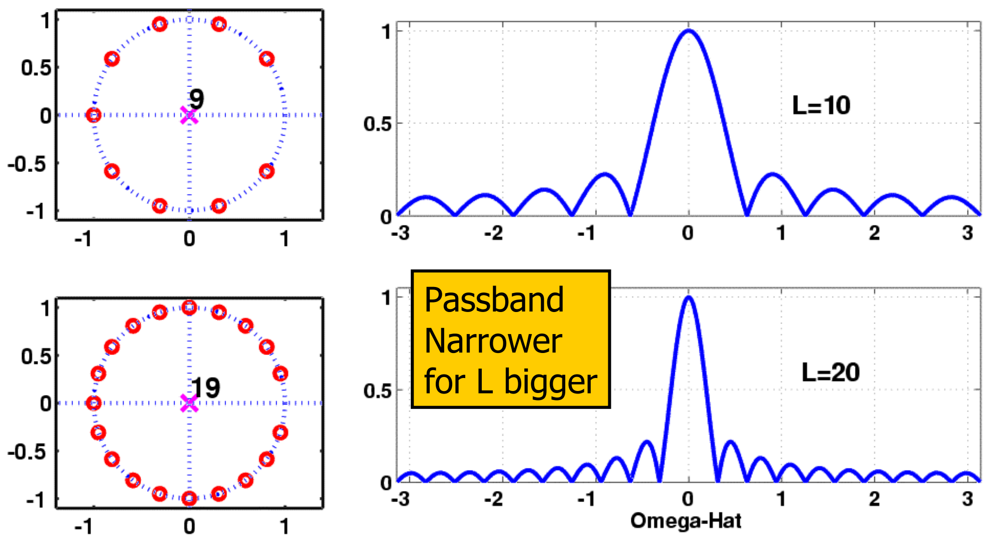
$$z^L - 1 = 0 \quad \Rightarrow \quad z^L = 1 = e^{j2\pi k}$$

$$z = e^{j(2\pi/L)k} \quad \text{for } k = 1, 2, \ldots L\text{-}1$$

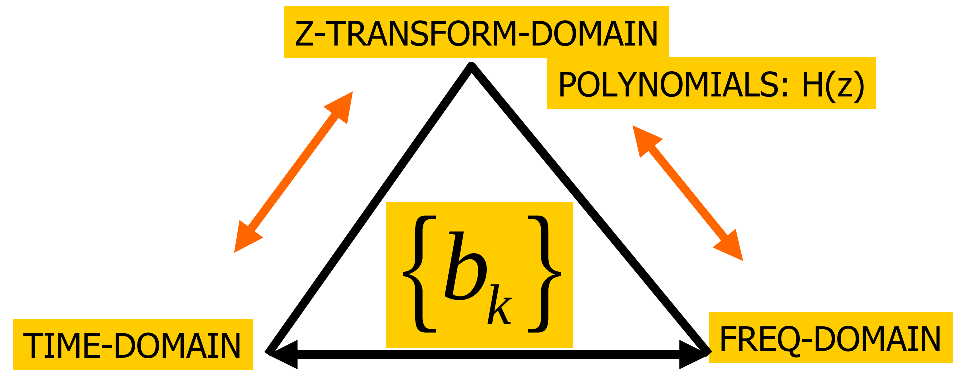
ZEROS on UNIT CIRCLE

(z-1) in denominator cancels k=0 term

FILTER DESIGN: CHANGE L



THREE DOMAINS



$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

LECTURE OBJECTIVES

INFINITE IMPULSE RESPONSE FILTERS

Define IIR Filters

Have FEEDBACK: PREVIOUS OUTPUTS

$$y[n] = \sum_{\ell=1}^{N} a_\ell y[n-\ell] + \sum_{k=0}^{M} b_k x[n-k]$$

- Show how to compute the output y[n]
  - FIRST-ORDER CASE (N=1)
  - Z-transform: Impulse Response h[n] <--> H(z)

LOGICAL THREAD

- FIND the IMPULSE RESPONSE, h[n]
  - INFINITELY LONG
  - **IIR** Filters

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

- EXPLOIT THREE DOMAINS:
  - Show Relationship for IIR:

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

ONE FEEDBACK TERM

ADD PREVIOUS OUTPUTS

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

PREVIOUS FEEDBACK

FIR PART of the FILTER FEED-FORWARD

- CAUSALITY
  - NOT USING FUTURE OUTPUTS or INPUTS

FILTER COEFFICIENTS

- ADD PREVIOUS OUTPUTS

$$y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]$$

FEEDBACK COEFFICIENT

SIGN CHANGE

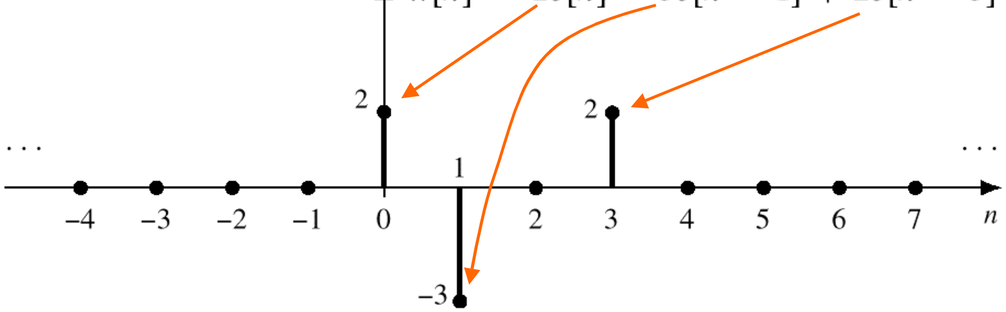
- MATLAB
  - `yy = filter([3,-2],[1,-0.8],xx)`

$$y[n] - 0.8y[n-1] = 3x[n] - 2x[n-1]$$

COMPUTE OUTPUT

$$y[n] = 0.8y[n-1] + 5x[n]$$

$$x[n] = 2\delta[n] - 3\delta[n-1] + 2\delta[n-3]$$



COMPUTE y[n]

- FEEDBACK DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 5x[n]$$

NEED y[-1] to get started

$$y[0] = 0.8y[-1] + 5x[0]$$

AT REST CONDITION

- y[n] = 0, for n<0
- BECAUSE x[n] = 0, for n<0

INITIAL REST CONDITIONS

1. The input must be assumed to be zero prior to some starting time n_0, i.e., $x[n] = 0$ for $n < n_0$. We say that such inputs are *suddenly applied.*
2. The output is likewise assumed to be zero prior to the starting time of the signal, i.e., $y[n] = 0$ for $n < n_0$. We say that the system is *initially at rest* if its output is zero prior to the application of a suddenly applied input.

COMPUTE y[0]

- THIS STARTS THE RECURSION:

With the initial rest assumption, $y[n] = 0$ for $n < 0$,

$$y[0] = 0.8y[-1] + 5(2) = 0.8(0) + 5(2) = 10$$

- SAME with MORE FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]$$

COMPUTE MORE y[n]

- CONTINUE THE RECURSION:

$$y[1] = 0.8y[0] + 5x[1] = 0.8(10) + 5(-3) = -7$$

$$y[2] = 0.8y[1] + 5x[2] = 0.8(-7) + 5(0) = -5.6$$

$$y[3] = 0.8y[2] + 5x[3] = 0.8(-5.6) + 5(2) = 5.52$$

$$y[4] = 0.8y[3] + 5x[4] = 0.8(5.52) + 5(0) = 4.416$$

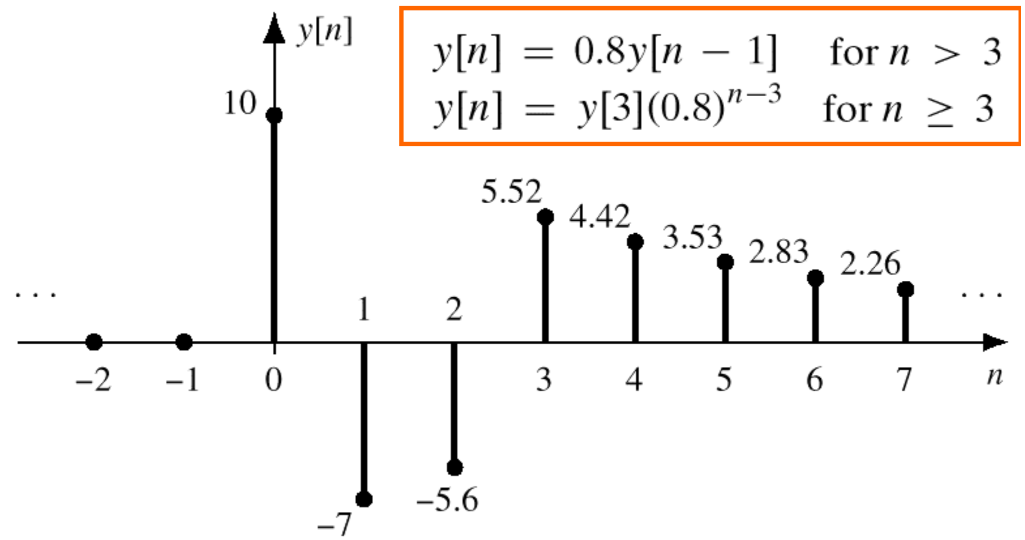
$$y[5] = 0.8y[4] + 5x[5] = 0.8(4.416) + 5(0) = 3.5328$$

$$y[6] = 0.8y[5] + 5x[6] = 0.8(3.5328) + 5(0) = 2.8262$$

PLOT y[n]

$$y[n] = 0.8y[n-1] \quad \text{for } n > 3$$

$$y[n] = y[3](0.8)^{n-3} \quad \text{for } n \geq 3$$



IMPULSE RESPONSE

$$h[n] = a_1 h[n-1] + b_0 \delta[n]$$

n	$n<0$	0	1	2	3	4
$\delta[n]$	0	1	0	0	0	0
$h[n-1]$	0	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$
$h[n]$	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$	$b_0(a_1)^4$

From this table it is obvious that the general formula is

$$h[n] = \begin{cases} b_0(a_1)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$h[n] = b_0 (a_1)^n u[n]$$

$$u[n] = 1, \quad \text{for } n \geq 0$$

IMPULSE RESPONSE

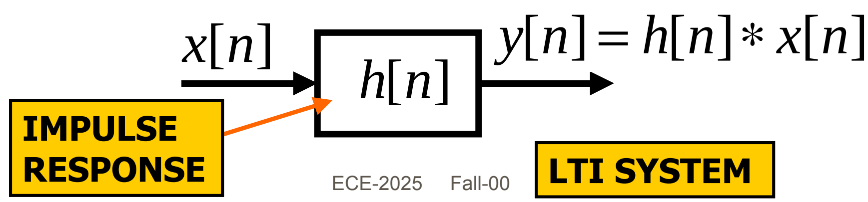
- DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 3x[n]$$

- Find h[n]

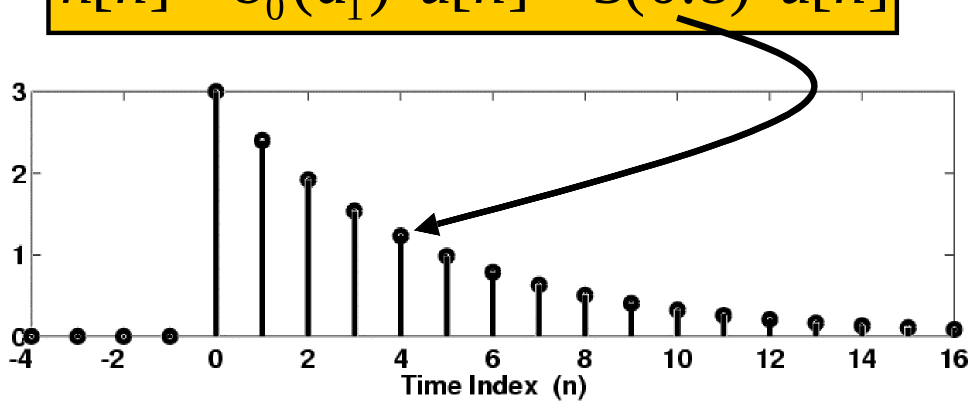
$$h[n] = 3(0.8)^n u[n]$$

- CONVOLUTION in TIME-DOMAIN



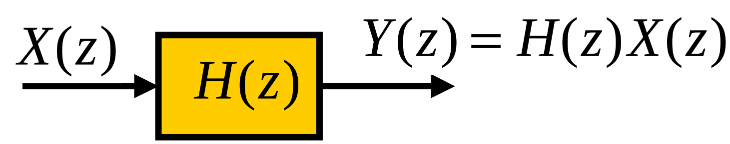
PLOT IMPULSE RESPONSE

$$h[n] = b_0 (a_1)^n u[n] = 3(0.8)^n u[n]$$

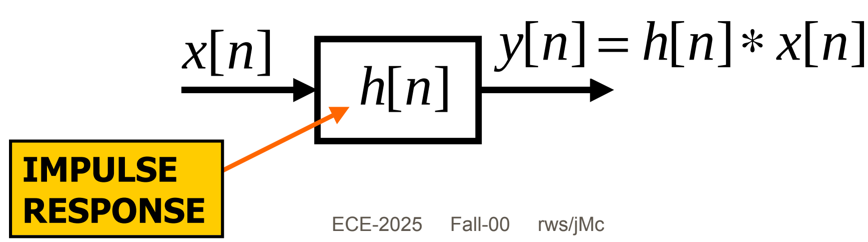


CONVOLUTION PROPERTY

- MULTIPLICATION of z-TRANSFORMS



- CONVOLUTION in TIME-DOMAIN



Infinite-Length Signal: h[n]

- POLYNOMIAL Representation

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

APPLIES to Any SIGNAL

- SIMPLIFY the SUMMATION

$$H(z) = \sum_{n=-\infty}^{\infty} b_0 (a_1)^n u[n] z^{-n} = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n}$$

Derivation of H(z)

- Recall Sum of Geometric Sequence:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

- Yields a COMPACT FORM

$$H(z) = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n$$

$$= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1|$$

H(z) = z-Transform{ h[n] }

- FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$h[n] = b_0 (a_1)^n u[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

H(z) = z-Transform{ h[n] }

- ANOTHER FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$h[n] = b_0 (a_1)^n u[n] + b_1 (a_1)^{n-1} u[n-1]$$

Z^{-1} is a shift

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}} + \frac{b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$