ECE 2025
SPRING '01

HOMEWORK # 8
SOLUTIONS

8.1) a) Obtain the difference equation describing the filter:

$$y[n] = \frac{1}{5} \sum_{k=0}^{4} x[n-k]$$

→ Use this equation to compute the values of $y[n]$ from $x[n]$.

b) Compute the output generated by each sinusoidal term:

$$4\cos\left(0.1\pi n + \frac{\pi}{2}\right) \Rightarrow 4\,|\mathcal{H}(0.1\pi)| \cdot \cos\left(0.1\pi n + \frac{\pi}{2} + \angle\mathcal{H}(0.1\pi)\right)$$

$$3\cos(0.4\pi n - \pi) \Rightarrow 3\,|\mathcal{H}(0.4\pi)| \cdot \cos\left(0.4\pi n - \pi + \angle\mathcal{H}(0.4\pi)\right)$$

→ Then add the two output terms to obtain the overall output generated by $x[n]$.

c) Compute $X(z) = \sum_{k=0}^{10} z^{-k}$

→ Compute $H(z) = \frac{1}{5} \sum_{k=0}^{4} z^{-k}$

→ $Y(z) = H(z) X(z)$

→ Compute the inverse z-transform of $Y(z)$ to get $y[n]$.

d) Use linearity and time-invariance:

$\delta[n] \Rightarrow h[n] = \frac{1}{5} \sum_{k=0}^{4} \delta[n-k]$

so $10\,\delta[n-50] \Rightarrow 10\,h[n-50]$

e) Use linearity:

$10\,\delta[n-50] \Rightarrow 10\,h[n-50]$

$4\cos(0.1\pi n + \frac{\pi}{2}) + 3\cos(0.4\pi n - \pi) \Rightarrow$ the output computed in part (b).

→ Add all the output terms to obtain the overall output generated by $x[n]$.

8.3) a) $h[n] = \delta[n] + 2\delta[n-2] + \delta[n-4]$

(by definition of impulse response, $h[n]$ is the output generated by $x[n] = \delta[n]$)

$$\mathcal{H}(\hat{\omega}) = 1 + 2e^{-j2\hat{\omega}} + e^{-j4\hat{\omega}}$$

$$H(z) = 1 + 2z^{-2} + z^{-4}$$

b) $y[n] = x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4]$

$$\mathcal{H}(\hat{\omega}) = 1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}} + e^{-j4\hat{\omega}}$$

$$H(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}$$

c) $$\mathcal{H}(\hat{\omega}) = \left(1 + \frac{e^{j2\hat{\omega}} + e^{-j2\hat{\omega}}}{2}\right)e^{-j3\hat{\omega}} =$$

$$= \frac{1}{2}e^{-j\hat{\omega}} + e^{-j3\hat{\omega}} + \frac{1}{2}e^{-j5\hat{\omega}}$$

$$H(z) = \frac{1}{2}z^{-1} + z^{-3} + \frac{1}{2}z^{-5}$$

$$h[n] = \frac{1}{2}\delta[n-1] + \delta[n-3] + \frac{1}{2}\delta[n-5]$$

$$y[n] = \frac{1}{2}x[n-1] + x[n-3] + \frac{1}{2}x[n-5]$$

d) $\mathcal{H}(\hat{\omega}) = 1 - 2e^{-j2\hat{\omega}} + e^{-j4\hat{\omega}} + e^{-j7\hat{\omega}}$

$h[n] = \delta[n] - 2\delta[n-2] + \delta[n-4] + \delta[n-7]$

$y[n] = x[n] - 2x[n-2] + x[n-4] + x[n-7]$

8.4) $\mathcal{H}(\hat{\omega}) = 1 + e^{-j2\hat{\omega}}$

Consider each term separately:

$10 \xrightarrow{C/D} 10 \xrightarrow{LTI} 10 \cdot \mathcal{H}(0) = 10 \cdot 2 = 20 \xrightarrow{D/C} 20$

$4\cos\left(4000\pi t - \frac{\pi}{8}\right)$:

$\omega = \pm 4000\pi \xrightarrow{C/D} \hat{\omega} = \frac{\omega}{f_s} = \pm \frac{\pi}{2}$

$\mathcal{H}\left(\pm\frac{\pi}{2}\right) = 1 + e^{\mp j 2\frac{\pi}{2}} = 1 + e^{\mp j\pi} = 0$

Therefore this term is removed by the filter.

$6\cos\left(11000\pi t - \frac{\pi}{3}\right) = 3e^{-j\frac{\pi}{3}}e^{j11000\pi t} + 3e^{j\frac{\pi}{3}}e^{-j11000\pi t}$

$$\omega = \pm 11000\,\pi \xrightarrow{C/D} \hat{\omega} = \pm \frac{11}{8}\pi$$

(*Warning*: These values of $\hat{\omega}$ are outside the $[-\pi, \pi]$ interval, therefore aliasing occurs during D/C conversion).

$$\mathcal{H}\left(\pm \frac{11}{8}\pi\right) = 1 + e^{\mp j 2 \frac{11}{8}\pi} = 1 + e^{\mp j \frac{11}{4}\pi} =$$

$$= 0.7654\, e^{\mp j\, 1.1781} = 0.7654\, e^{\mp j \frac{3\pi}{8}}$$

$$3 e^{-j\frac{\pi}{3}} e^{j\frac{11}{8}\pi n} \xrightarrow{LTI} 2.296\, e^{-j\frac{17}{24}\pi} e^{j\frac{11}{8}\pi n} \xrightarrow{D/C}$$

$$\xrightarrow{D/C} 2.296\, e^{-j\frac{17}{24}\pi} e^{-j 5000\pi t}$$

(because of aliasing, $\omega = (\hat{\omega} - 2\pi) f_s = -5000\pi$)

$$3 e^{j\frac{\pi}{3}} e^{-j\frac{11}{8}\pi n} \xrightarrow{LTI} 2.296\, e^{j\frac{17}{24}\pi} e^{-j\frac{11}{8}\pi n} \xrightarrow{D/C}$$

$$\xrightarrow{D/C} 2.296\, e^{j\frac{17}{24}\pi} e^{j 5000\pi t}$$

(because of aliasing, $\omega = (\hat{\omega} + 2\pi) f_s = 5000\pi$).

$$y(t) = 20 + 2.296\, e^{j(5000\pi t + \frac{17}{24}\pi)} + 2.296\, e^{-j(5000\pi t + \frac{17}{24}\pi)} =$$

$$= 20 + 4.592 \cos\left(5000\pi t + \frac{17}{24}\pi\right).$$

8.5) a) $h[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3] =$
$$= \sum_{k=0}^{3} \delta[n-k]$$

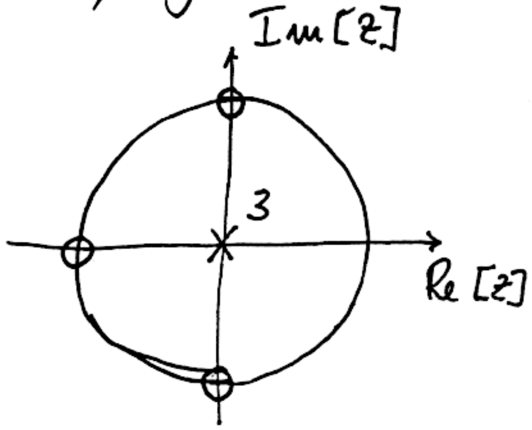
b) $$H(z) = 1 + z^{-1} + z^{-2} + z^{-3} = \sum_{k=0}^{3} z^{-k} = \frac{1-z^{-4}}{1-z^{-1}} =$$
$$= \frac{z^4 - 1}{z^3(z-1)}$$ (using the formula: $\sum_{k=0}^{N} x^k = \frac{1-x^{N+1}}{1-x}$).

c) $H(z) = 0 \Rightarrow z^4 - 1 = 0 \Rightarrow z = \pm 1, \pm j$

(the equation $z^N = 1$ has N solutions:
$$z = e^{j\frac{2k\pi}{N}}, \quad k = 0, 1, 2, \dots, N-1.$$
These values of z are called the <u>N-th roots of unity</u>).

However, the denominator of $\frac{z^4-1}{z^3(z-1)}$ also vanishes for $z = 1$, giving an indeterminate value $\frac{0}{0}$.

Using the expression $H(z) = \sum_{k=0}^{3} z^{-k}$, we obtain $H(1) = 4 \neq 0$. Therefore the zeros of $H(z)$ are $-1, \pm j$.



8.6) a) $$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{3} e^{-jk\hat{\omega}} = \frac{1-e^{-j4\hat{\omega}}}{1-e^{-j\hat{\omega}}}$$

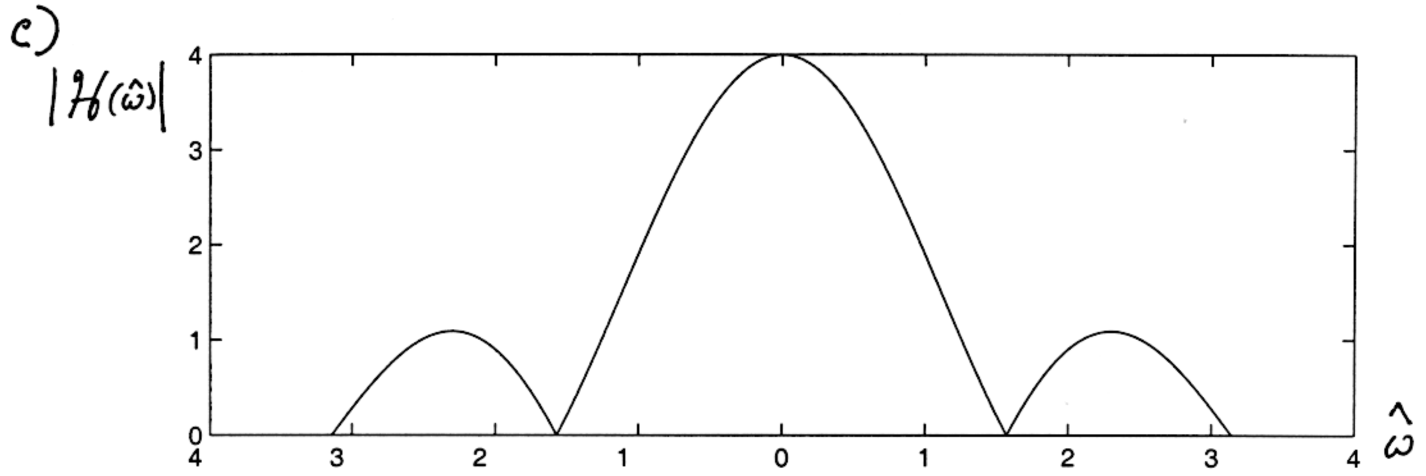
(using again $\sum_{k=0}^{N} x^k = \frac{1-x^{N+1}}{1-x}$, or simply set $z = e^{j\hat{\omega}}$ in the expression for $H(z)$ obtained in 8.5)

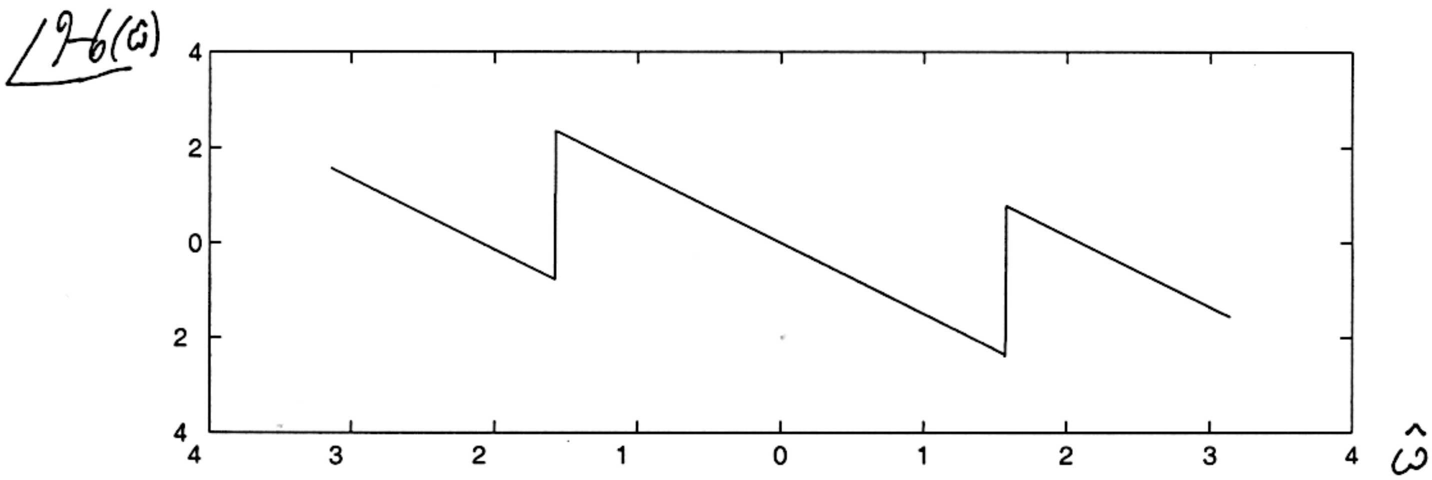
b) $$\mathcal{H}(\hat{\omega}) = \frac{e^{-j2\hat{\omega}}}{e^{-j\frac{\hat{\omega}}{2}}} \frac{e^{j2\hat{\omega}}}{e^{j\frac{\hat{\omega}}{2}}} \frac{1-e^{-j4\hat{\omega}}}{1-e^{-j\hat{\omega}}} =$$

$$= e^{-j\frac{3\hat{\omega}}{2}} \frac{e^{j2\hat{\omega}} - e^{-j2\hat{\omega}}}{e^{j\frac{\hat{\omega}}{2}} - e^{-j\frac{\hat{\omega}}{2}}} =$$

$$= e^{-j\frac{3\hat{\omega}}{2}} \frac{2j \sin 2\hat{\omega}}{2j \sin \frac{\hat{\omega}}{2}} = \frac{\sin(2\hat{\omega})}{\sin \frac{\hat{\omega}}{2}} e^{-j\frac{3\hat{\omega}}{2}}$$

c)





d) The sinusoid is removed by the filter if $\mathcal{H}(\hat{\omega}_0) = 0$. This means we have to find those values of $\hat{\omega}_0$, $0 < \hat{\omega}_0 \leq \pi$ such that $\sin(2\hat{\omega}_0) = 0$ ($\hat{\omega}_0 = 0$ is excluded because $\mathcal{H}(0) = 4$, from 8.5(c)).

Since $\sin\theta = 0 \Leftrightarrow \theta = m\pi$, $m = 0, \pm 1, \pm 2, \ldots$

we must have $2\hat{\omega}_0 = m\pi$, i.e. $\hat{\omega}_0 = \frac{\pi}{2}$ or

$\hat{\omega}_0 = \pi$ (other values for m give values for $\hat{\omega}_0$

that are outside the specified interval).

For those values of $\hat{\omega}_0$, the output is:

$y[n] = \mathcal{H}(0)\cdot 1 = 4.$

8.7) $H(z) = (1-z^{-1})(1+z^{-1})(1+z^{-2}) = (1-z^{-2})(1+z^{-2}) =$

$= 1 - z^{-4}$

a) $y[n] = x[n] - x[n-4]$

b) $h[n] = \delta[n] - \delta[n-4]$

c) $X(z) = 2z^{-1} + 2z^{-3} - 2z^{-4} = 2z^{-1}(1 + z^{-2} - z^{-3})$

$H(z)X(z) = 2z^{-1}(1-z^{-4})(1+z^{-2}-z^{-3}) =$

$= 2z^{-1} + 2z^{-3} - 2z^{-4} - 2z^{-5} - 2z^{-7} + 2z^{-8} = Y(z)$

$y[n] = 2\delta[n-1] + 2\delta[n-3] - 2\delta[n-4] +$

$-2\delta[n-5] - 2\delta[n-7] + 2\delta[n-8]$

d) From the factored form of $H(z)$, we see that the zeros of $H(z)$ are $z = 1, -1, j, -j$.

These are all on the unit circle, and correspond to $\hat{\omega} = 0, \pi, \frac{\pi}{2}, -\frac{\pi}{2}$ respectively. For these values of $\hat{\omega}$ the output will be zero.