EE-2025 Fall-2000

Lecture 6
Fourier Series Coefficients
11-Sept-00

Web-CT Info

- Check the Bulletin Board for msgs
- **Get Notes on Fourier Series**
  - 17 pages, posted to WebCT
  - Replacement for pp. 62-66 in Chapter 3
- Prob Set #2 due This Week
  - Solution will be posted Thurs evening

Quiz #1 Info

- Quiz #1 on 18-Sept (Monday)
  - Calculator OK, and one page of notes
  - Coverage: HW #1 and #2
- Old Quizzes & Problems are linked via WebCT: "Word from Previous Semesters"

Lab Info

- Lab #2 Report
  - Turn in during your lab time
  - Write-up section 4 on "multipath"
  - Finish INSTRUCTOR VERIFICATION in Lab
  - ERRATA ? ALWAYS Check Bulletin Board
- Lab #3 was posted Friday
  - Learn some Music Notation

LECTURE

EE-2025 Georgia Tech

Lecture 6

Fourier Series Coefficients

READING ASSIGNMENTS

- This Lecture:
  - **Notes on Fourier Series**
    - 17 pages, posted to WebCT
    - Replace pp 62-66 in Chapter 3
- Other Reading:
  - Next Lecture: Chap. 4 on Sampling

LECTURE OBJECTIVES

- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi k/T_0)t}\,dt$$

- **ANALYSIS** via Fourier Series
  - For PERIODIC signals: **x(t+T) = x(t)**
  - **SPECTRUM from the Fourier Series**

HISTORY

- Jean Baptiste Joseph Fourier
  - 1807 thesis (memoir)
    - On the Propagation of Heat in Solid Bodies
  - Heat !
  - Napoleonic era
- http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html

Joseph Fourier

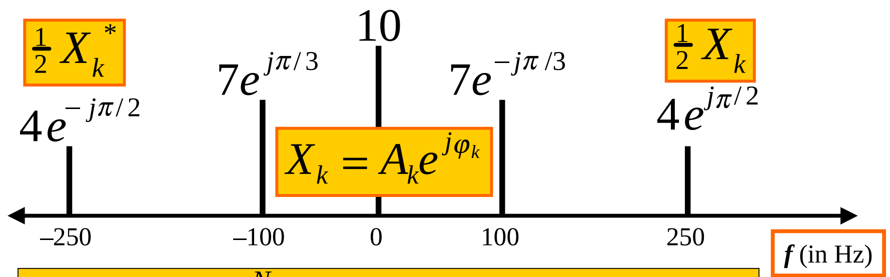
lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Find out more at:
http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fourier.html

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \tfrac{1}{2} X_k e^{j2\pi f_k t} + \tfrac{1}{2} X_k^* e^{-j2\pi f_k t} \right\}$$

Harmonic Signal

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$x(t) = X_0 + \sum_{k=1}^{N} \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \sum_{k=1}^{N} \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$\omega_0 = \frac{2\pi k}{T_0} = \left(\frac{2\pi}{T_0}\right)k = 2\pi(f_0)k$$

Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

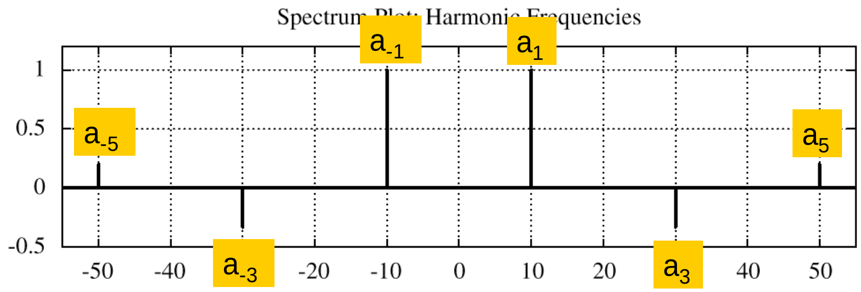
$$a_k = \tfrac{1}{2} X_k = \tfrac{1}{2} A_k e^{j\varphi_k}$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$X_k = A_k e^{j\varphi_k}$$

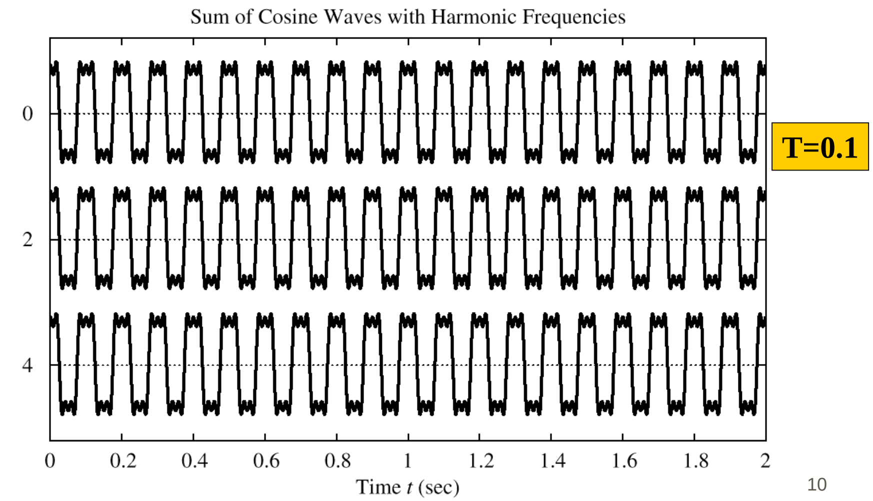
COMPLEX AMPLITUDE

Harmonic Signal (3 Freqs)



a_k is the complex amplitude for kf_0

Harmonic Signal (3 Freqs)



SYNTHESIS vs. ANALYSIS

- SYNTHESIS
  - Easy
  - Given (ω_k, A_k, ϕ_k) create x(t)
- Synthesis can be HARD
  - Synthesize Speech so that it sounds good
- ANALYSIS
  - Hard
  - Given x(t), extract (ω_k, A_k, ϕ_k)
  - How many?
  - Need algorithm for computer

STRATEGY

- ANALYSIS
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - The answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}\,dt$$

Fourier Series Integral

- HOW do you determine a_k from x(t) ?

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi k t/T_0}\,dt$$

FUNDAMENTAL FREQ: $f_0 = 1/T_0$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt \quad \text{(DC Component)}$$

$$a_{-k} = a_k^* \quad \text{when } x(t) \text{ is real}$$

ORTHOGONALITY of exp(j)

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j2\pi m t/T_0}\,dt = \frac{T_0}{-j2\pi m}e^{-j2\pi m t/T_0}\Big|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m}\left(e^{-j2\pi m} - 1\right)$$

$$\int_0^{T_0} e^{-jm\omega_0 t}\,dt = 0$$

$$\omega_0 = \frac{2\pi}{T_0}$$

ORTHOGONALITY of exp(j)

- INTEGRATE over ONE PERIOD

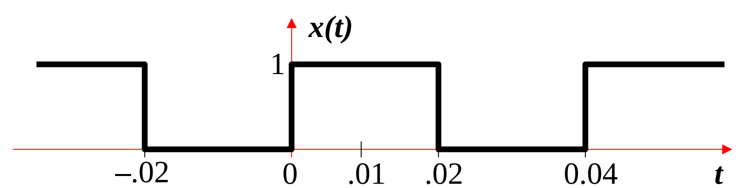
$$\frac{1}{T_0}\int_0^{T_0} e^{j2\pi \ell t/T_0}\, e^{-j2\pi k t/T_0}\,dt = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$$

$$\frac{1}{T_0}\int_0^{T_0} e^{j2\pi(\ell - k)t/T_0}\,dt$$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \tfrac{1}{2}T_0 \\ 0 & \tfrac{1}{2}T_0 \le t < T_0 \end{cases}$$

$$\text{for } T_0 = 0.04 \text{ sec:}$$



FS for a SQUARE WAVE

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt \qquad (k \neq 0)$$

$$a_k = \frac{1}{.04}\int_0^{.02} 1e^{-j2\pi kt/.04}dt = \frac{1}{.04(-j\pi k/.02)}\, e^{-j\pi kt/.02}\Big|_0^{.02}$$

$$= \frac{1}{(-2j\pi k)}\left(e^{-j\pi k} - 1\right) = \frac{1-(-1)^k}{j2\pi k}$$

DC Coefficient, a_0

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt \qquad (k = 0)$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0}(\text{AREA})$$

$$a_0 = \tfrac{1}{.04}\int_0^{.02} dt = \tfrac{1}{.04}(.02-0) = \tfrac{1}{2}$$

Fourier Coefficients a_k

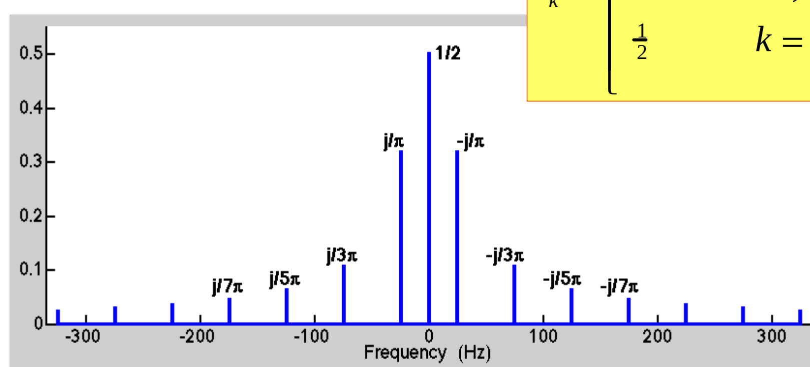
- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

$$a_k = \frac{1-e^{-j\pi k}}{j2\pi k} = \begin{cases} \dfrac{1}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \tfrac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

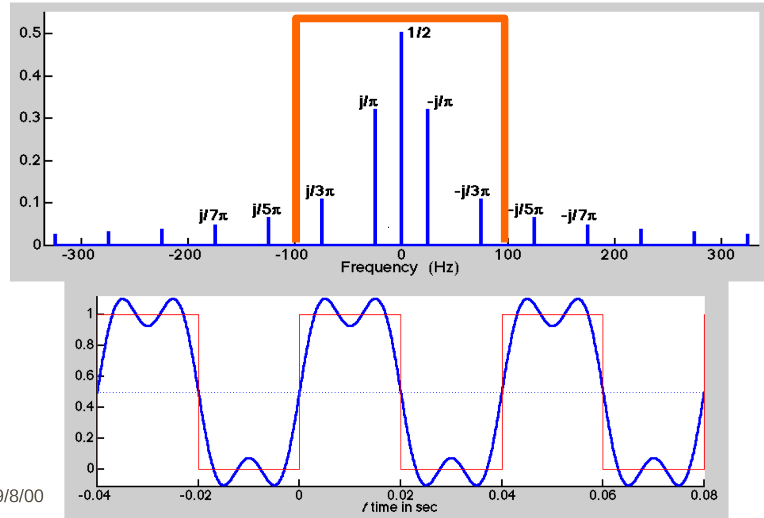
$$\omega_0 = 2\pi/(0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \dfrac{-j}{\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \tfrac{1}{2} & k = 0 \end{cases}$$



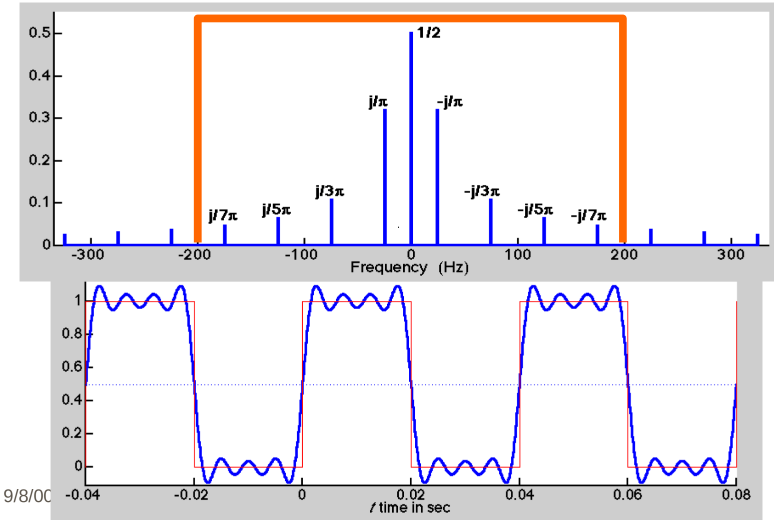
Synthesis: 1st & 3rd Harmonics

$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos\left(2\pi(25)t - \tfrac{\pi}{2}\right) + \frac{2}{3\pi}\cos\left(2\pi(75)t - \tfrac{\pi}{2}\right)$$



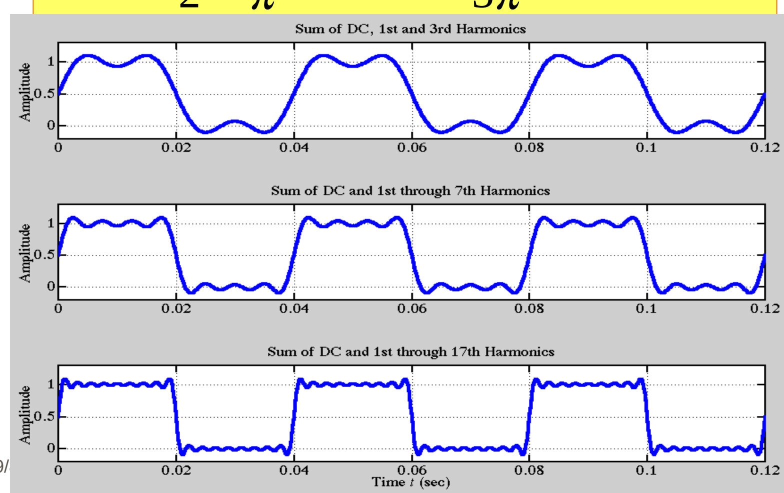
Synthesis: up to 7th Harmonic

$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos\left(50\pi t - \tfrac{\pi}{2}\right) + \frac{2}{3\pi}\sin(150\pi t) + \frac{2}{5\pi}\sin(250\pi t) + \frac{2}{7\pi}\sin(350\pi t)$$



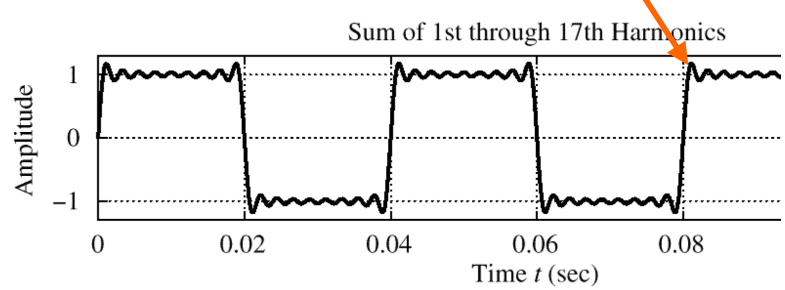
Fourier Synthesis

$$x_N(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \dots$$



Gibbs' Phenomenon

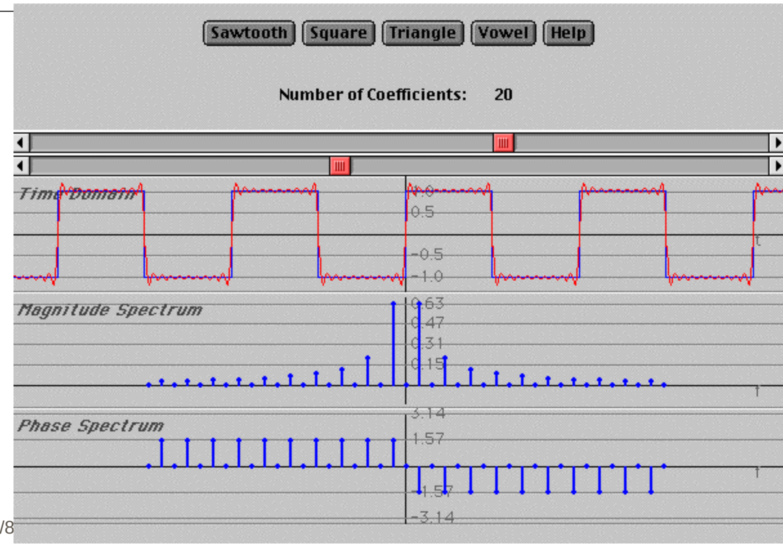
- Convergence at DISCONTINUITY of x(t)
  - There is always an overshoot
  - 9% for the Square Wave case



Fourier Series Demo

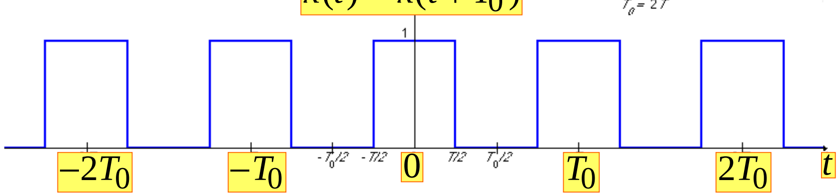
- Fourier Series Java Applet
  - Greg Slabaugh
    - Interactive
    - http://users.ece.gatech.edu/~slabaugh/java/fourier/fourier.html

Fourier Series Java Applet



General Periodic Signals

$x(t) = x(t + T_0)$



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

Fourier Synthesis

Fundamental Freq.

$$\omega_0 = 2\pi / T_0 = 2\pi f_0$$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

Fourier Analysis