

EE-2025

Fall-2000

Lecture 23

IIR Filters: Feedback &  $H(z)$

27-Nov-00

## Info: HW and Labs

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- Reading Assignment:
  - Chapter 8 in DSP First
- Prob Set #11 - due this week
  - Prob Set #12 next week
- Lab #11 due this week of 27-Nov
  - Lab #12 due next week (4—7 Dec)
  - ALL Labs must be turned in by 8-Dec-00

11/26/00

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2

## Final Exam Info

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- Calendar: Final Exam(s)
  - Period 1, Monday, 11-Dec @ 8 am
  - Period 8, Wed, 13-Dec @ 11:30am
    - \*\*Must take exam with your assigned section\*\*
- Report **CONFLICTS immediately !!!!**
  - E.g., 3 exams in one day
- Reviews will be held on Sunday & Tuesday
  - 7pm in ECE Auditorium

## Grades

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- Quiz #3 Results
  - Raw Median = 70; average = 66.5
  - Adjusted Median = 78
- Check your own grades for accuracy
- Cumulative Averages are now posted
  - No HW dropped yet.
  - Max = 70 (at this moment)
    - Final will be 25 (max)
    - Recitation Points will be 5 (max)

11/26/00

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3

11/26/00

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4

## Lecture 23

IIR Filters: Feedback  
and  $H(z)$ 

## READING ASSIGNMENTS

## ■ This Lecture:

- Chapter 8, pp. 249-263

## ■ Other Reading:

- Recitation: Ch. 8, pp. 261-272

- POLES & ZEROS

- Next Lecture: Chapter 8, pp. 269-282

## LECTURE OBJECTIVES

## ■ INFINITE IMPULSE RESPONSE FILTERS

- Define **IIR** DIGITAL Filters

- Have **FEEDBACK**: use PREVIOUS OUTPUTS

$$y[n] = \sum_{\ell=1}^N a_{\ell} y[n-\ell] + \sum_{k=0}^M b_k x[n-k]$$

- Show how to compute the output  $y[n]$

- FIRST-ORDER CASE ( $N=1$ )

- Z-transform: Impulse Response  $h[n] \leftrightarrow H(z)$

## Quick Review: L-pt Averager

$$y[n] = \sum_{k=0}^{L-1} \frac{1}{L} x[n-k]$$

IMPULSE RESPONSE

$$h[n] = \sum_{k=0}^{L-1} \frac{1}{L} \delta[n-k]$$

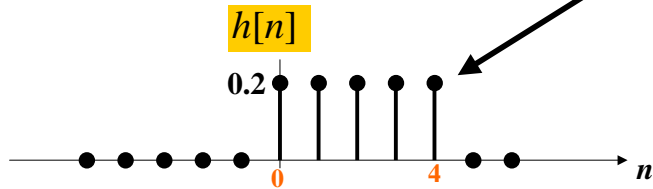
SYSTEM FUNCTION

$$H(z) = \sum_{n=0}^{L-1} \frac{1}{L} z^{-n}$$

# MATH FORMULA for h[n]

Use **SHIFTED** IMPULSES to write h[n]

$$h[n] = \frac{1}{5} \delta[n] + \frac{1}{5} \delta[n-1] + \frac{1}{5} \delta[n-2] + \frac{1}{5} \delta[n-3] + \frac{1}{5} \delta[n-4]$$



$$\{b_k\} = \left\{ \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \right\}$$

# LTI: Convolution

Output = Convolution of x[n] & h[n]

NOTATION:  $y[n] = x[n] * h[n]$

Here is the FIR case:

$$y[n] = \sum_{k=0}^M h[k] x[n-k]$$

FINITE LIMITS

Same as  $b_k$

FINITE LIMITS

# L-pt RUNNING AVG H(z)

$$H(z) = \sum_{k=0}^{L-1} \frac{1}{L} z^{-k} = \frac{1 - z^{-L}}{L(1 - z^{-1})} = \frac{z^L - 1}{Lz^{L-1}(z - 1)}$$

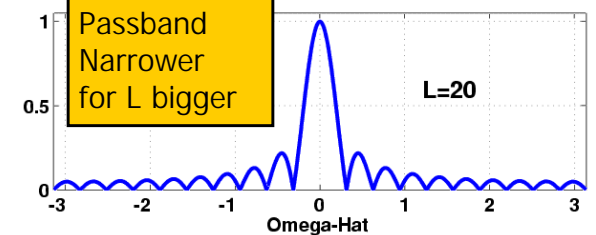
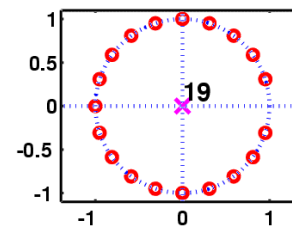
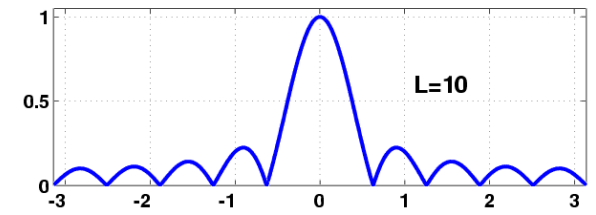
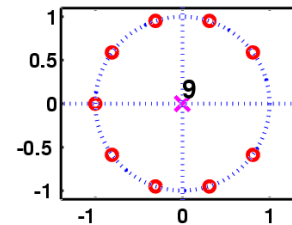
$$z^L - 1 = 0 \Rightarrow z^L = 1 = e^{j2\pi k}$$

$$z = e^{j(2\pi/L)k} \text{ for } k = 1, 2, \dots, L-1$$

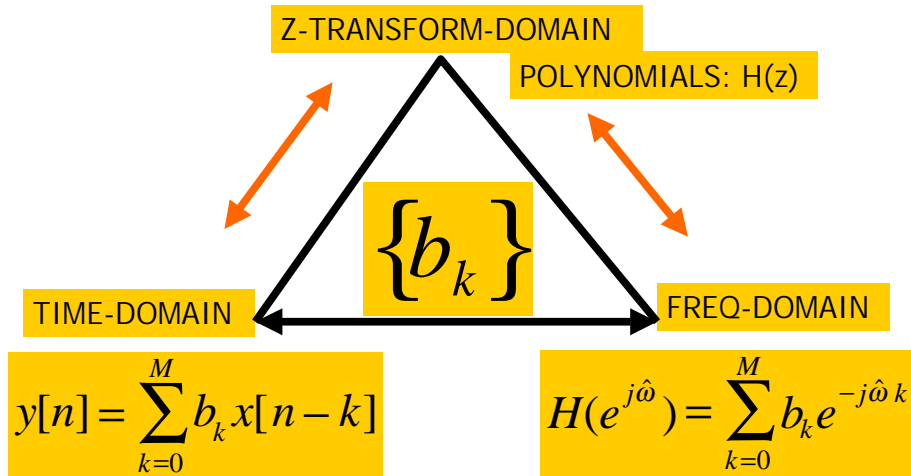
ZEROS on UNIT CIRCLE

(z-1) in denominator cancels k=0 term

# FILTER DESIGN: CHANGE L



# THREE DOMAINS



# LECTURE OBJECTIVES

## INFINITE IMPULSE RESPONSE FILTERS

- Define IIR Filters
- Have **FEEDBACK**: PREVIOUS OUTPUTS

$$y[n] = \sum_{\ell=1}^N a_{\ell} y[n-\ell] + \sum_{k=0}^M b_k x[n-k]$$

- Show how to compute the output  $y[n]$ 
  - FIRST-ORDER CASE (N=1)
  - Z-transform: Impulse Response  $h[n] \leftrightarrow H(z)$

# LOGICAL THREAD

## FIND the IMPULSE RESPONSE, $h[n]$

- INFINITELY LONG
- IIR** Filters

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

## EXPLOIT THREE DOMAINS:

- Show Relationship for IIR:

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

# ONE FEEDBACK TERM

## ADD PREVIOUS OUTPUTS

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$



## CAUSALITY

- NOT USING **FUTURE** OUTPUTS or INPUTS

## FILTER COEFFICIENTS

- ADD PREVIOUS OUTPUTS

$$y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]$$

FEEDBACK COEFFICIENT

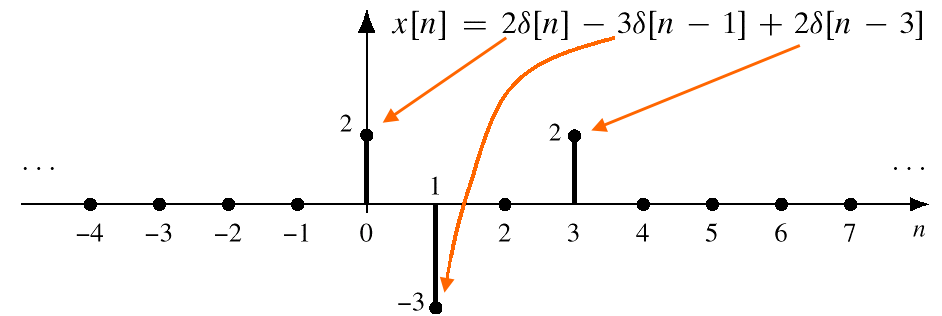
SIGN CHANGE

- MATLAB

```
yy = filter([3,-2],[1,-0.8],xx)
```

## COMPUTE OUTPUT

$$y[n] = 0.8y[n-1] + 5x[n]$$



## COMPUTE $y[n]$

- FEEDBACK DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 5x[n]$$

- NEED  $y[-1]$  to get started

$$y[0] = 0.8y[-1] + 5x[0]$$

## AT REST CONDITION

- $y[n] = 0$ , for  $n < 0$
- BECAUSE  $x[n] = 0$ , for  $n < 0$

### INITIAL REST CONDITIONS

1. The input must be assumed to be zero prior to some starting time  $n_0$ , i.e.,  $x[n] = 0$  for  $n < n_0$ . We say that such inputs are *suddenly applied*.
2. The output is likewise assumed to be zero prior to the starting time of the signal, i.e.,  $y[n] = 0$  for  $n < n_0$ . We say that the system is *initially at rest* if its output is zero prior to the application of a suddenly applied input.

## COMPUTE $y[0]$

### THIS STARTS THE RECURSION:

With the initial rest assumption,  $y[n] = 0$  for  $n < 0$ ,

$$y[0] = 0.8y[-1] + 5(2) = 0.8(0) + 5(2) = 10$$

### SAME with MORE FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^2 b_k x[n-k]$$

## COMPUTE MORE $y[n]$

### CONTINUE THE RECURSION:

$$y[1] = 0.8y[0] + 5x[1] = 0.8(10) + 5(-3) = -7$$

$$y[2] = 0.8y[1] + 5x[2] = 0.8(-7) + 5(0) = -5.6$$

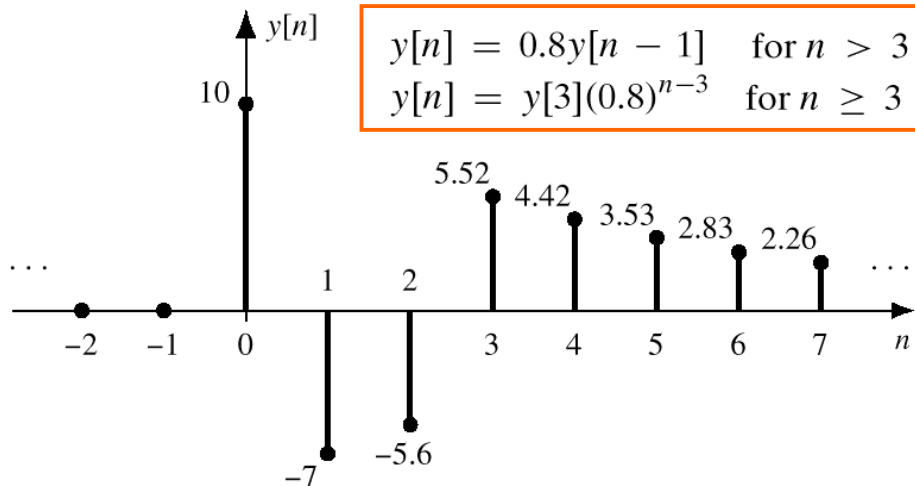
$$y[3] = 0.8y[2] + 5x[3] = 0.8(-5.6) + 5(2) = 5.52$$

$$y[4] = 0.8y[3] + 5x[4] = 0.8(5.52) + 5(0) = 4.416$$

$$y[5] = 0.8y[4] + 5x[5] = 0.8(4.416) + 5(0) = 3.5328$$

$$y[6] = 0.8y[5] + 5x[6] = 0.8(3.5328) + 5(0) = 2.8262$$

## PLOT $y[n]$



## IMPULSE RESPONSE

$$h[n] = a_1 h[n-1] + b_0 \delta[n]$$

$n$	$n < 0$	0	1	2	3	4
$\delta[n]$	0	1	0	0	0	0
$h[n-1]$	0	0	$b_0$	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$
$h[n]$	0	$b_0$	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$	$b_0(a_1)^4$

From this table it is obvious that the general formula is

$$h[n] = \begin{cases} b_0(a_1)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$h[n] = b_0(a_1)^n u[n]$$

$$u[n] = 1, \text{ for } n \geq 0$$

## IMPULSE RESPONSE

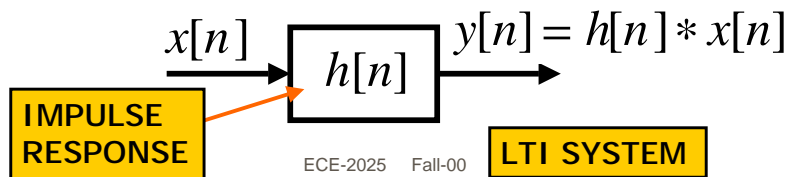
- DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 3x[n]$$

- Find  $h[n]$

$$h[n] = 3(0.8)^n u[n]$$

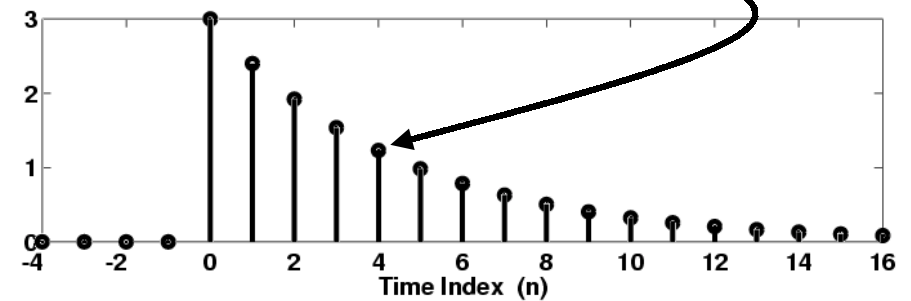
- CONVOLUTION** in TIME-DOMAIN



21

## PLOT IMPULSE RESPONSE

$$h[n] = b_0(a_1)^n u[n] = 3(0.8)^n u[n]$$



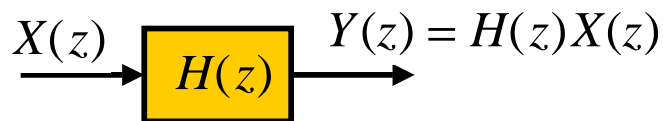
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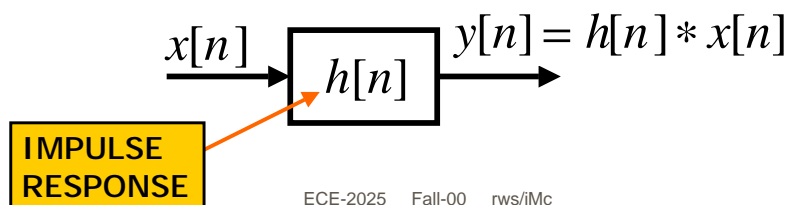
22

## CONVOLUTION PROPERTY

- MULTIPLICATION of z-TRANSFORMS



- CONVOLUTION in TIME-DOMAIN



23

## Infinite-Length Signal: $h[n]$

- POLYNOMIAL Representation

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

APPLIES to Any SIGNAL

- SIMPLIFY the SUMMATION

$$H(z) = \sum_{n=-\infty}^{\infty} b_0(a_1)^n u[n]z^{-n} = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n}$$

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24

## Derivation of H(z)

- Recall Sum of Geometric Sequence:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

- Yields a COMPACT FORM

$$\begin{aligned} H(z) &= b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n \\ &= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a| \end{aligned}$$

## H(z) = z-Transform{ h[n] }

- FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$h[n] = b_0 a^n u[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

## H(z) = z-Transform{ h[n] }

- ANOTHER FIRST-ORDER IIR FILTER:

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$h[n] = b_0 a^n u[n] + b_1 a^{n-1} u[n-1]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}} + \frac{b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$