# EE-2025 Fall-00

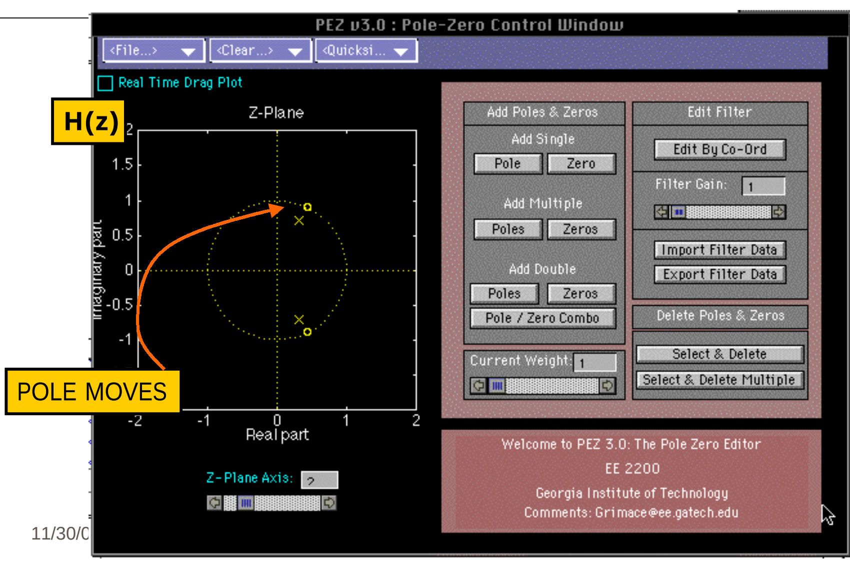
Lecture 24
H(z) & Frequency Response
1-Dec-00

# Final Exam Info

- Calendar: Final Exam(s)
  - Period 1, Monday, 11-Dec @ 8 am
  - Period 8, Wed, 13-Dec @ 11:30am
- **Report CONFLICTS immediately !!!!**
  - E.g., 3 exams in one day
- Reviews will be held on Sunday & Tuesday
  - 7pm in ECE Auditorium



# PeZ GUI for MATLAB






# ZZZZZ-Transform

# EE-2025 Georgia Tech

Lecture 24

IIF Filters: H(z) & Frequency Response

# READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, pp. 263-279
- Other Reading:
  - Recitation: Ch. 8, pp. 261-272
    - POLES & ZEROS



# LECTURE OBJECTIVES

- SYSTEM FUNCTION: H(z)
- H(z) has **POLES** and ZEROS
- FREQUENCY RESPONSE of IIR
  - Get H(z) first

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$



# IIR FILTER REVIEW

- ADD PREVIOUS OUTPUTS

$$y[n] = 0.8y[n-1] + 3x[n]$$

FEEDBACK COEFFICIENT

SIGN CHANGE

- MATLAB
  - `yy = filter([3],[1,-0.8],xx)`

$$H(z) = \frac{3}{1-0.8z^{-1}}$$

## IMPULSE RESPONSE

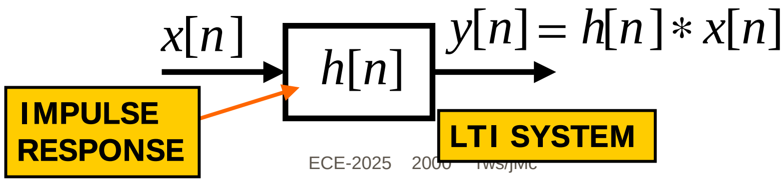
- DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 3x[n]$$

- Find h[n]

$$h[n] = 3(0.8)^n u[n]$$

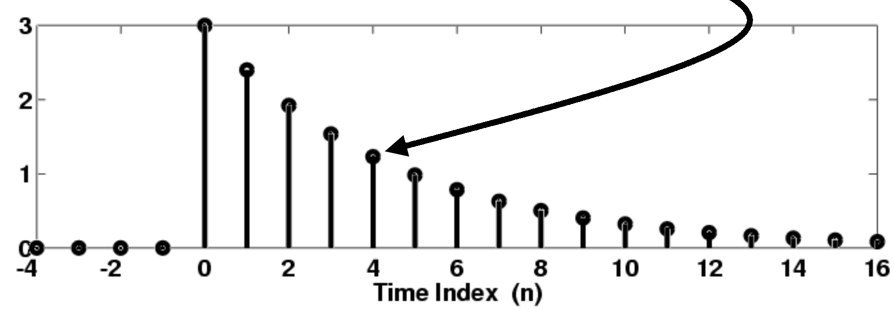
- **CONVOLUTION** in TIME-DOMAIN





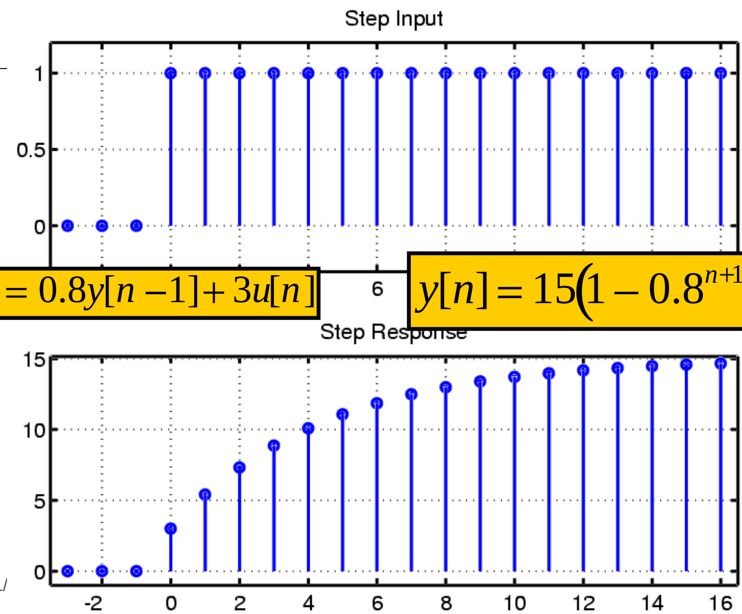

## PLOT IMPULSE RESPONSE

$$h[n] = b_0(a_1)^n u[n] = 3(0.8)^n u[n]$$






## PLOT STEP RESPONSE




$$y[n] = 0.8y[n-1] + 3u[n]$$

$$y[n] = 15\left(1 - 0.8^{n+1}\right)u[n]$$



## THREE DOMAINS

**Use H(z) to get Freq. Response**

$$z = e^{j\hat{\omega}}$$

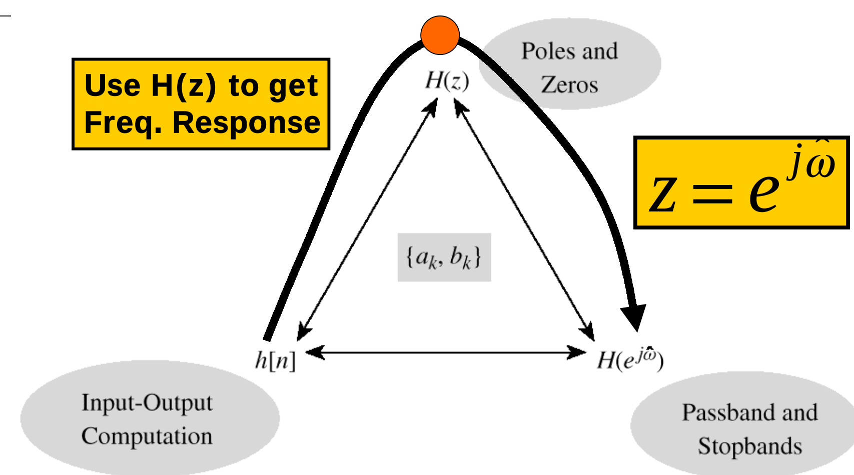



**Figure 8.13** Relationship among the $n$-, $z$-, and $\hat{\omega}$-domains. The filter coefficients $\{a_k,\ b_k\}$ play a central role.

## First-Order Transform Pair

- GEOMETRIC SEQUENCE:

$$h[n] = ba^n u[n] \quad \leftrightarrow \quad H(z) = \frac{b}{1 - az^{-1}}$$

- USE KNOWN TRANSFORM PAIR:

$$h[n] = ba^n u[n] = 3(0.8)^n u[n]$$

$$H(z) = \sum_n 3(0.8)^n u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} 3(0.8)^n z^{-n} = \frac{3}{1 - 0.8z^{-1}}$$



## DELAY PROPERTY of X(z)

- DELAY in TIME<-->Multiply X(z) by $z^{-1}$

$$x[n] \leftrightarrow X(z)$$

$$x[n-1] \leftrightarrow z^{-1} X(z)$$

$$\text{Proof:} \quad \sum_{n=-\infty}^{\infty} x[n-1] z^{-n} = \sum_{\ell=-\infty}^{\infty} x[\ell] z^{-(\ell+1)}$$

$$= z^{-1} \sum_{\ell=-\infty}^{\infty} x[\ell] z^{-\ell} = z^{-1} X(z)$$



## Z-Transform of IIR Filter

- DERIVE the SYSTEM FUNCTION H(z)
  - Use **DELAY** PROPERTY

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)$$

**EASIER with DELAY PROPERTY**

*Time delay of $n_0$ samples multiplies the z-transform by $z^{-n_0}$*

$$x[n - n_0] \quad \Longleftrightarrow \quad z^{-n_0} X(z)$$



## SYSTEM FUNCTION

- DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]$$

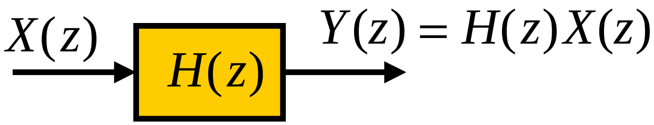
- **READ** the FILTER COEFFS:

**H(z)**

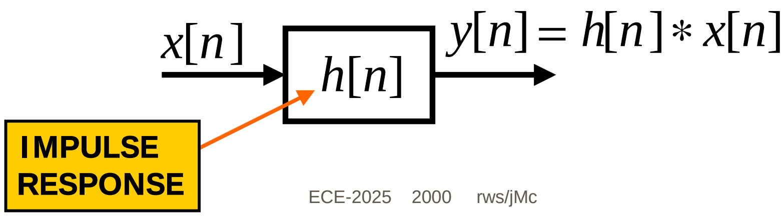
$$Y(z) = \left( \frac{3 - 2z^{-1}}{1 - 0.8z^{-1}} \right) X(z)$$

## CONVOLUTION PROPERTY

- **MULTIPLICATION** of z-TRANSFORMS




- **CONVOLUTION** in TIME-DOMAIN






## POLES & ZEROS

- ROOTS of NUMERATOR & DENOMINATOR

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} \rightarrow H(z) = \frac{b_0 z + b_1}{z - a_1}$$

$$b_0 z + b_1 = 0 \quad \Rightarrow z = -\frac{b_1}{b_0}$$

**ZERO: H(z)=0**

$$z - a_1 = 0 \quad \Rightarrow z = a_1$$

**POLE: H(z) -> inf**



## EXAMPLE: Poles & Zeros

- VALUE of H(z) at POLES is **INFINITE**

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

$$H(-1) = \frac{2 + 2(-1)}{1 - 0.8(-1)} = 0$$

**ZERO at z= -1**

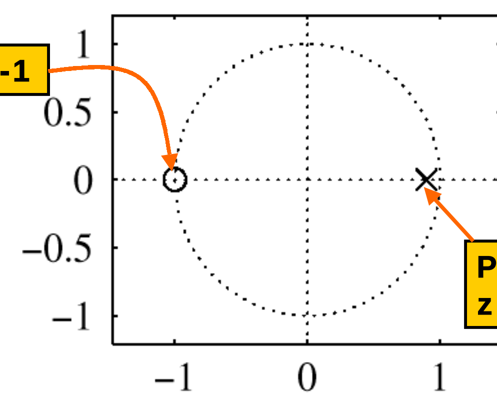
$$H(\tfrac{4}{5}) = \frac{2 + 2(\frac{4}{5})}{1 - 0.8(\frac{4}{5})} = \frac{\frac{7}{5}}{0} \rightarrow \infty$$

**POLE at z=0.8**



## POLE-ZERO PLOT

**ZERO at z = -1**




$$\frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

**POLE at z=0.8**

# FREQUENCY RESPONSE

- SYSTEM FUNCTION: H(z)
- H(z) has DENOMINATOR
- FREQUENCY RESPONSE of IIR
  - We have H(z)

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$



# FREQUENCY RESPONSE

- EVALUATE on the UNIT CIRCLE

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}} = \frac{b_0 + b_1 e^{-j\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}}}$$



# FREQ. RESPONSE FORMULA

$$H(z) = \frac{1}{1-0.8z^{-1}} \rightarrow H(e^{j\hat{\omega}}) = \frac{1}{1-0.8e^{-j\hat{\omega}}}$$

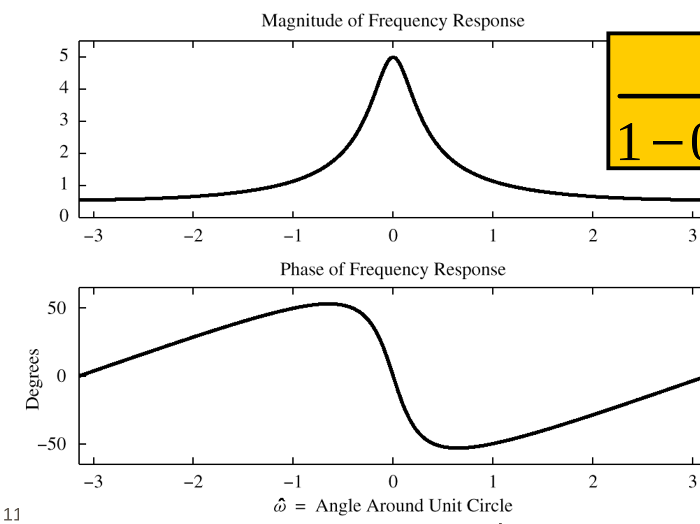
$$\left|H(e^{j\hat{\omega}})\right|^2 = \left|\frac{1}{1-0.8e^{-j\hat{\omega}}}\right|^2 = \frac{1}{1-0.8e^{-j\hat{\omega}}} \cdot \frac{1}{1-0.8e^{j\hat{\omega}}}$$

$$\frac{1}{1+0.64-0.8e^{-j\hat{\omega}}-0.8e^{j\hat{\omega}}} = \frac{1}{1.64-1.6\cos\hat{\omega}}$$

$$@\hat{\omega}=0, \quad \left|H(e^{j\hat{\omega}})\right|^2 = \frac{1}{0.04} = 25 \qquad @\hat{\omega}=\pi ?$$



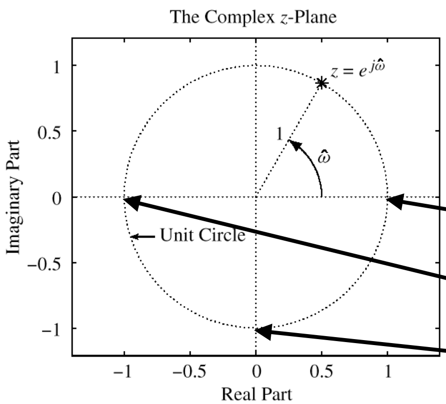
# FREQ. RESPONSE from H(z)




$$\frac{1}{1-0.8e^{-j\hat{\omega}}}$$

# UNIT CIRCLE

- MAPPING BETWEEN $z$ and $\hat{\omega}$




$$z = e^{j\hat{\omega}}$$

$$z = 1 \quad \leftrightarrow \quad \hat{\omega} = 0$$

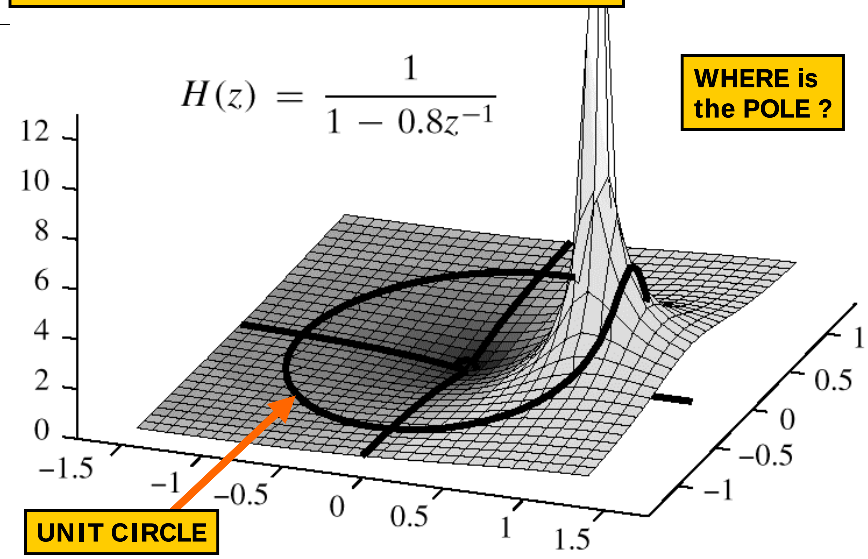
$$z = -1 \quad \leftrightarrow \quad \hat{\omega} = \pm\pi$$

$$z = \pm j \quad \leftrightarrow \quad \hat{\omega} = \pm\tfrac{1}{2}\pi$$



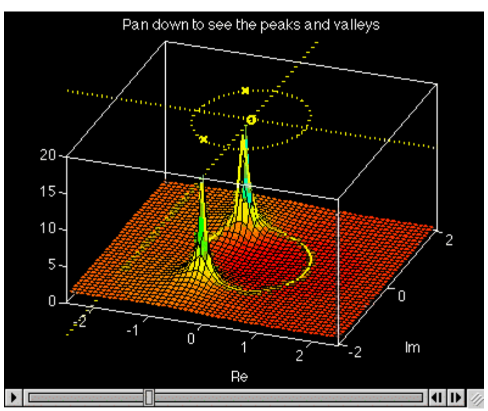
# 3-D VIEWPOINT: EVALUTE H(z) EVERYWHERE

$$H(z) = \frac{1}{1 - 0.8z^{-1}}$$




# MOVIE for H(z) in 3-D

- POLES to H(z) to Frequency Reponse
  - TWO POLES SHOWN






# THREE DOMAINS

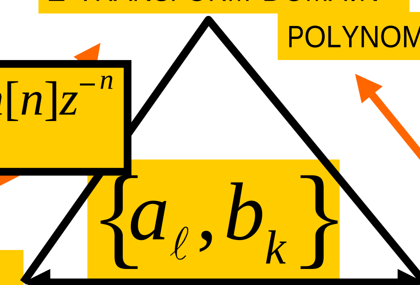
**Use H(z) to get Freq. Response**

Z-TRANSFORM-DOMAIN

POLYNOMIALS: H(z)

$$H(z) = \sum_{n} h[n]z^{-n}$$

$$z = e^{j\hat{\omega}}$$



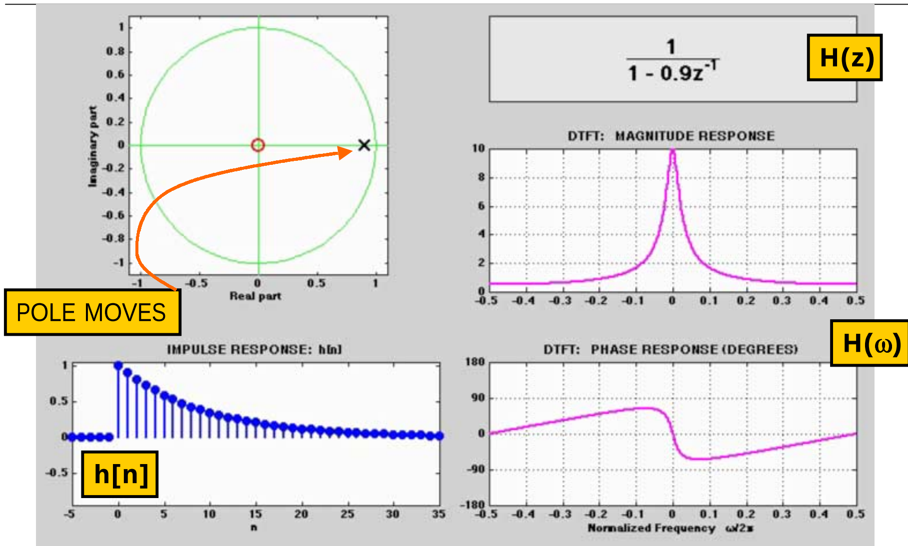
$$\{a_\ell, b_k\}$$

TIME-DOMAIN

FREQ-DOMAIN

$$y[n] = a_1 y[n-1] + \sum_{k=0}^{M} b_k x[n-k]$$

# 3 DOMAINS MOVIE: IIR




# POP QUIZ

- Given: $H(z) = \dfrac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$
- Find the Impulse Response, h[n]
- Find the output, y[n]
  - When $x[n] = \cos(0.25\pi n)$



# POP QUIZ: Invert Z

- Given:

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}} = \frac{2}{1 - 0.8z^{-1}} + \frac{2z^{-1}}{1 - 0.8z^{-1}}$$

- Find the Impulse Response, h[n]
  - Use the DELAY PROPERTY

$$h[n] = 2(0.8)^n u[n] + 2(0.8)^{n-1} u[n-1]$$



# SINUSOIDAL RESPONSE

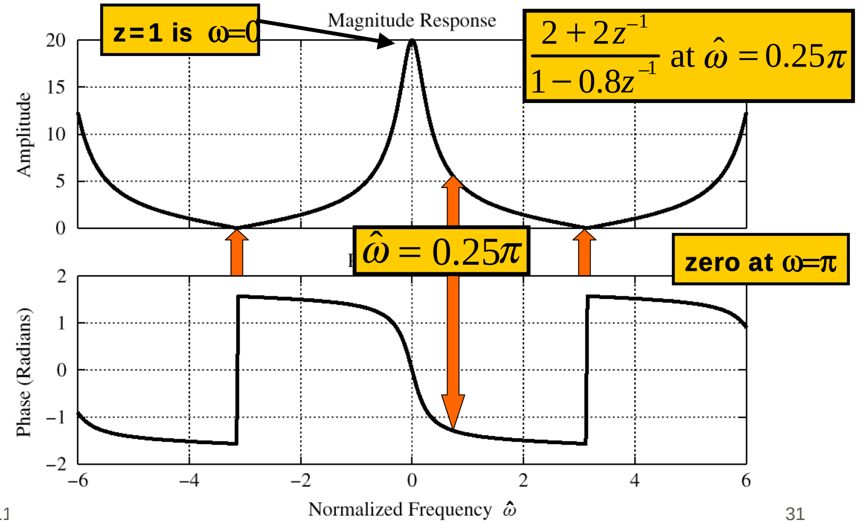
- x[n] = SINUSOID => y[n] is SINUSOID
- Get MAGNITUDE & PHASE from H(z)

$$\text{if } x[n] = e^{j\hat{\omega} n}, \text{ then}$$

$$y[n] = \mathcal{H}(\hat{\omega}) e^{j\hat{\omega} n}$$

$$\mathcal{H}(\hat{\omega}) = H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$$

# Evaluate FREQ. RESPONSE








# POP QUIZ: Eval Freq. Resp.

- Given: $H(z)=\frac{2+2z^{-1}}{1-0.8z^{-1}}$
- Find output, y[n], when $x[n]=\cos(0.25\pi n)$
  - Evaluate at $z=e^{j0.25\pi}$

$$H(z)=\frac{2+2(\frac{\sqrt{2}}{2}+j\frac{\sqrt{2}}{2})}{1-0.8e^{-j0.25\pi}}=5.182e^{-j1.309}$$

$$y[n]=5.182\cos(0.25\pi n-0.417\pi)$$