EE-2200 Spring-99

Lecture 5

Harmonics:

Fourier Series Coefficients

16-April-99

Web-CT Info

- Check the Bulletin Board for msgs
- Lectures are being posted
- Old Quizzes & Problems are linked
  - Quiz #1 on 26-April (Monday)
- Prob Set #2 due Friday, 16-April
- On-Line Homework will continue

Lab Info

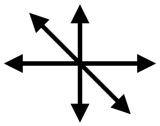
- Lab #2 Report
  - ERRORS !!! ALWAYS Check Bulletin Board
  - Turn in during your lab time
  - Write-up sections 4 and 5
  - Include INSTRUCTOR VERIFICATION
- Lab #3 will be posted on Monday
  - Music Notation will be needed

READING ASSIGNMENTS

- This Lecture:
  - Chapter 3, pp. 57-68
- Other Reading:
  - Notes on Fourier Series
    - (3 pages posted to WebCT)
  - Next Lecture: Chap. 3, pp. 68-77

Problem Solving Skills

- **Math Formula**
  - Sum of Cosines
  - (A_k, ω_k, ϕ_k)
- **Recorded Signals**
  - Speech
  - Music
  - No simple formula



- **Plots & Sketches**
  - x(t) versus t
  - Spectrum
- **MATLAB**
  - Numerical
  - Computation
  - Plotting lists of numbers

LECTURE OBJECTIVES

- Signals with **HARMONIC** Frequencies
  - Add Sinusoids with $f_k = kf_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

- **ANALYSIS** via Fourier Series
  - For **PERIODIC** signals: $x(t+T) = x(t)$

HISTORY

- Jean Baptiste Joseph Fourier
  - 1807 thesis (memoir)
    - On the Propagation of Heat in Solid Bodies
  - Heat !
  - Napoleonic era
- http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html

Joseph Fourier

lived from 1768 to 1830

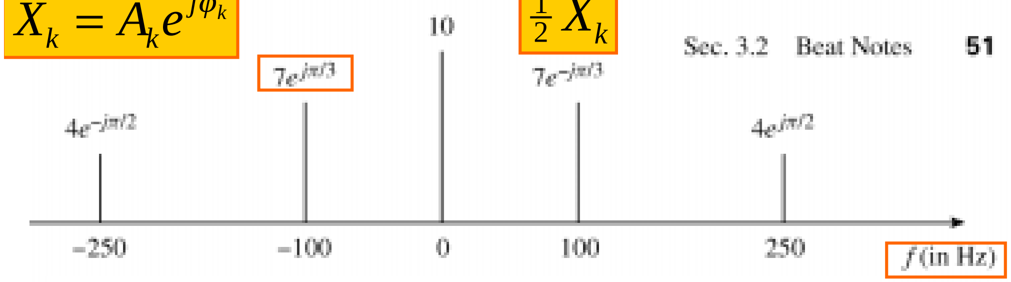
Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Find out more at:
http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fourier.html

FREQUENCY DIAGRAM

- Recall Complex Amplitude vs. Freq

$X_k = A_k e^{j\varphi_k}$ $\quad \frac{1}{2} X_k$



$$x(t) = 10 + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2)$$

Summary: GENERAL FORM

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \phi_k)$$

$$x(t) = X_0 + \sum_{k=1}^{N} \Re e\left\{X_k e^{j2\pi f_k t}\right\}$$

$X_k = A_k e^{j\phi_k}$

frequency is f_k.

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \frac{X_k}{2} e^{j2\pi f_k t} + \frac{X_k^*}{2} e^{-j2\pi f_k t} \right\}$$

PERIODIC SIGNALS

- Repeat every T secs
  - Definition

$$x(t) = x(t+T)$$

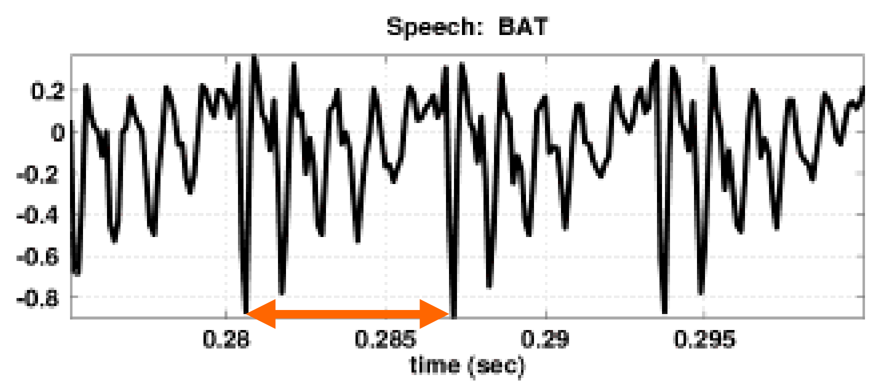
  - Example:

$$x(t) = \cos^2(3t) \qquad T = ?$$

  - Speech can be “quasi-periodic”

Speech Signal: BAT

- Nearly Periodic in the Vowel Region
  - Period is (Approximately) T = 0.0065 sec



Period of Complex Exponential

$$x(t) = e^{j\omega_0 t}$$

$$x(t+T) = x(t)\ ?$$

Definition: Period is T

$$e^{j\omega_0 (t+T)} = e^{j\omega_0 t}$$

$$e^{j2\pi k} = 1$$

$$\Rightarrow e^{j\omega_0 T} = 1 \quad \Rightarrow \omega_0 T = 2\pi k$$

$$\omega_0 = \frac{2\pi k}{T} = \left(\frac{2\pi}{T}\right)k$$

k = integer

HARMONIC SIGNAL

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$f_k = k\, f_0$$

$$f_0 = \text{fundamental frequency}$$

$$T_0 = \text{fundamental Period}$$

$$f_0 = \frac{1}{T_0}$$

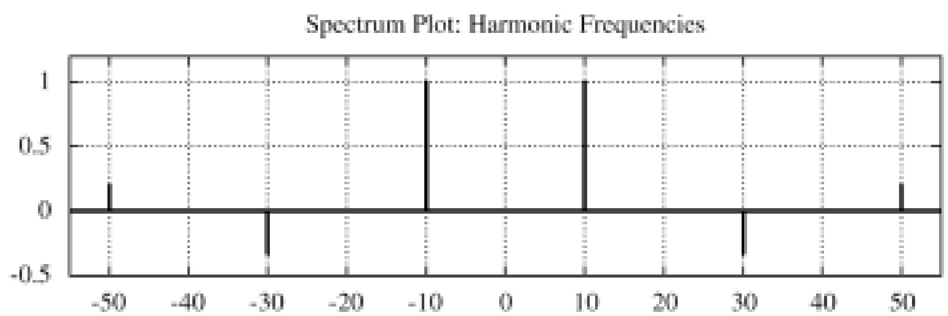
Harmonic Signal Spectrum

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$X_k = A_k e^{j\varphi_k}$$

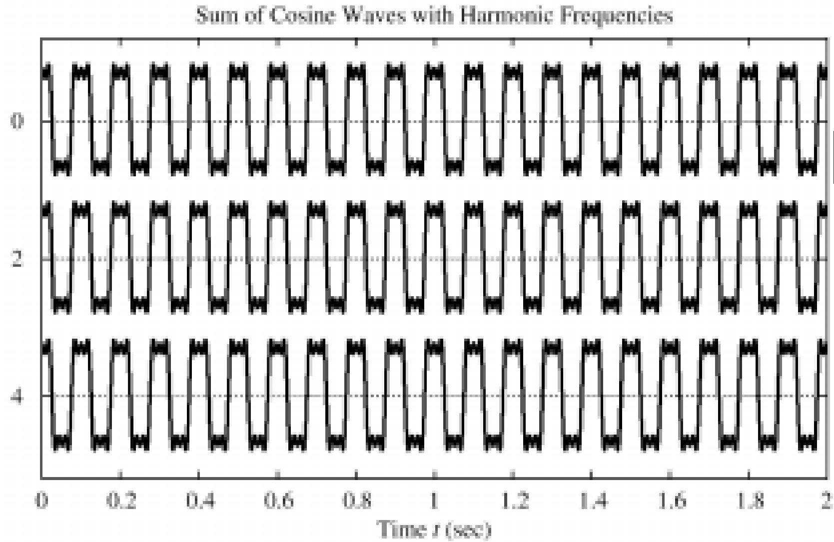
$$x(t) = X_0 + \sum_{k=1}^{N} \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \sum_{k=1}^{N} \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t}$$

Harmonic Signal (3 Freqs)

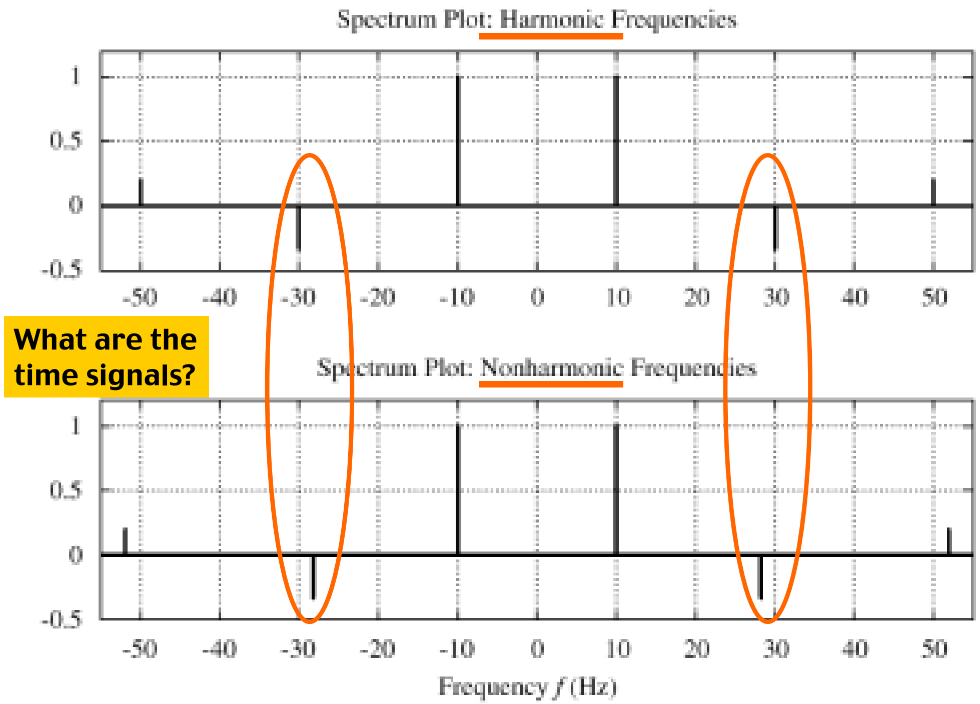


What is the fundamental frequency?

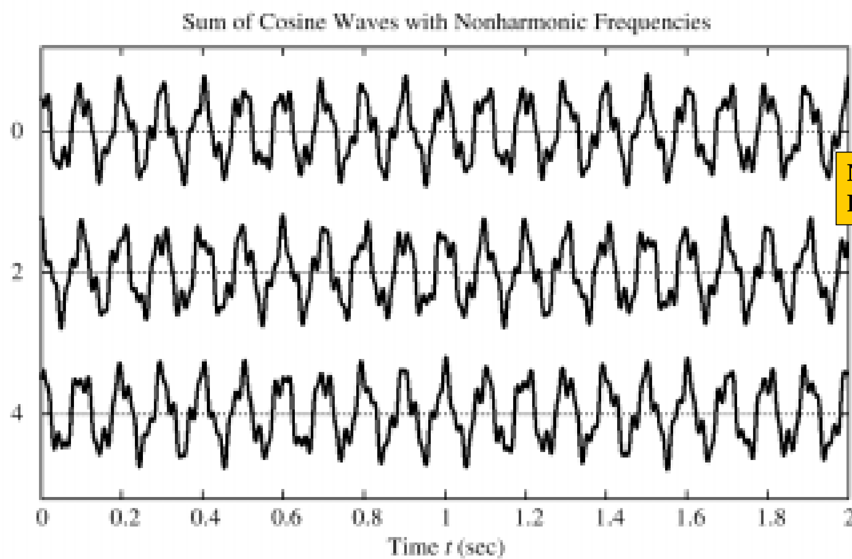
Harmonic Signal (3 Freqs)



T=0.1



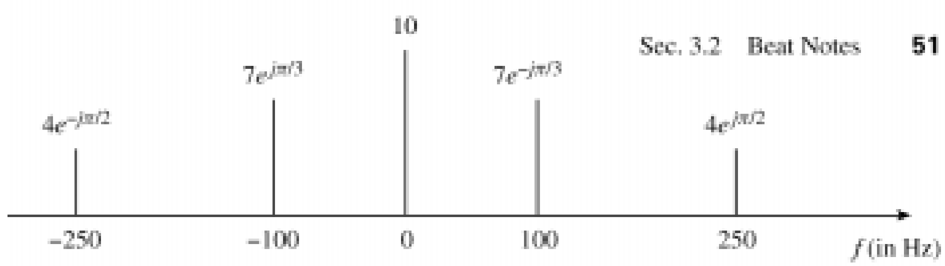
NON-Harmonic Signal



NOT PERIODIC

PERIOD from SPECTRUM

- Add the spectrum components:



What is the PERIOD for the signal x(t)?

Multiples of a Fundamental Frequency

- **Frequencies:**
  - −250 Hz (k=−5)
  - −100 Hz (k=−2)
  - 0 Hz (k=0)
  - 100 Hz (k=+2)
  - 250 Hz (k=+5)
- **Amplitude & Phase**
  - 4 $-\pi/2$
  - 7 $+\pi/3$
  - 10 0
  - 7 $-\pi/3$
  - 4 $+\pi/2$

FUNDAMENTAL FREQUENCY = 50 Hz
How do you find f_0 ?
Determine GCD: Greatest Common Divisor

GCD(100,250)

Example: Synthetic Vowel

- What is the Fundamental Frequency ?

f_k (Hz)	X_k	Mag	Phase (rad)
200	$(771 + j12202)$	12,226	1.508
400	$(-8865 + j28048)$	29,416	1.876
500	$(48001 - j8995)$	48,836	−0.185
1600	$(1657 - j13520)$	13,621	−1.449
1700	$4723 + j0$	4723	0

Table 3.1: Complex amplitudes for harmonic signal that approximates the vowel sound "ah".

Vowel Waveform (sum of all 5 components)

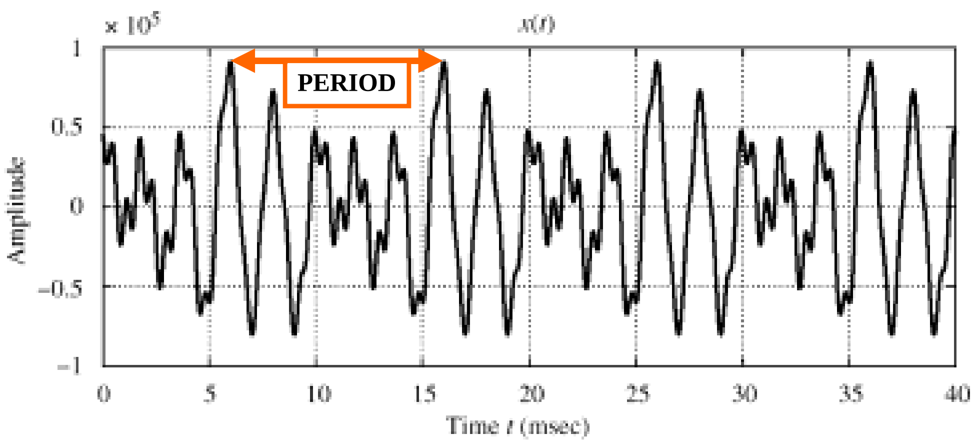


Figure 3.11 Sum of all of the terms in (3.3.4). Note that the period is 10 msec, which equals $1/f_0$.

SYNTHESIS vs. ANALYSIS

- **SYNTHESIS**
  - Easy
  - Given (ω_k, A_k, ϕ_k) create x(t)
- **Synthesis can be HARD**
  - Synthesize Speech so that it sounds good
- **ANALYSIS**
  - Hard
  - Given x(t), extract (ω_k, A_k, ϕ_k)
  - How many?
  - Need algorithm for computer

Fourier Series Expansion

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$X_k = A_k e^{j\varphi_k}$$

COMPLEX AMPLITUDE

$$x(t) = X_0 + \sum_{k=1}^{N} \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \sum_{k=1}^{N} \tfrac{1}{2} X_{-k} e^{-j2\pi k f_0 t}$$

$X_{-k} = X_k^*$ when $x(t)$ is real

Fourier Series Integral

- **HOW do you determine X_k from x(t) ?**

$$X_k = \frac{2}{T_0} \int_0^{T_0} x(t) e^{-j2\pi kt/T_0}\, dt$$

FUNDAMENTAL FREQ: $f_0 = 1/T_0$

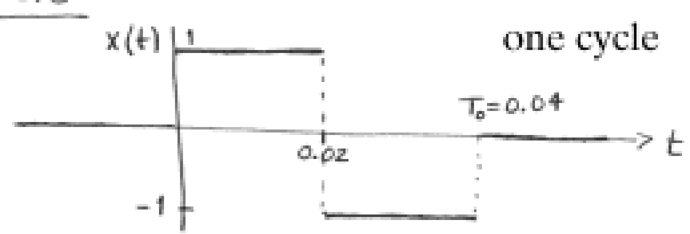
$$X_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\, dt$$

SQUARE WAVE

$$x(t) = \begin{cases} 1 & 0 \le t < \tfrac{1}{2}T_0 \\ -1 & \tfrac{1}{2}T_0 \le t < T_0 \end{cases} \qquad (3.4.4)$$

Draw a plot of the square wave defined in (3.4.4) for $T_0 = 0.04$ sec.

Ex 3.3



FS for a SQUARE WAVE

$$X_k = \frac{2}{T_0} \int_0^{\frac{1}{2}T_0} (1) e^{-j2\pi kt/T_0}\, dt + \frac{2}{T_0} \int_{\frac{1}{2}T_0}^{T_0} (-1) e^{-j2\pi kt/T_0}\, dt$$

which can be manipulated as follows:[5]

$$X_k = \frac{2}{T_0} \frac{e^{-j2\pi k(\frac{1}{2}T_0)/T_0} - e^{-j2\pi k(0)/T_0}}{-j2\pi k/T_0} + \frac{(-2)}{T_0} \frac{(e^{-j2\pi k T_0/T_0} - e^{-j2\pi k(\frac{1}{2}T_0)/T_0})}{-j2\pi k/T_0}$$

$$= \frac{e^{-j\pi k} - 1}{-j\pi k} + \frac{e^{-j\pi k} - e^{-j2\pi k}}{-j\pi k}$$

$$= \frac{2 - 2e^{-j\pi k}}{j\pi k} = \frac{2(1 - (-1)^k)}{j\pi k}$$

[5] We use the fact that $e^{-j2\pi k} = 1$ when k is an integer.

observe that the average value of this signal is zero, so $X_0 = 0$.

$$X_k = \begin{cases} \dfrac{4}{j\pi k} & k = \pm 1, \pm 3, \pm 5, \ldots \\ 0 & k = 0, \pm 2, \pm 4, \pm 6, \ldots \end{cases} \tag{3.4.5}$$

The magnitude of these coefficients is shown in Fig. 3.12. The phase angles are $-\pi/2$ for $k > 0$, and $\pi/2$ for $k < 0$. Note that if $f_0 = 1/T_0 = 25$ Hz, only the frequencies at ± 25, ± 75, ± 125, etc. are in the spectrum.

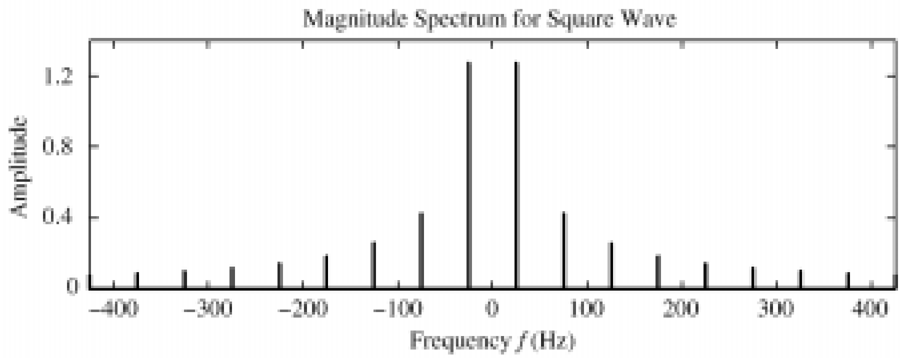
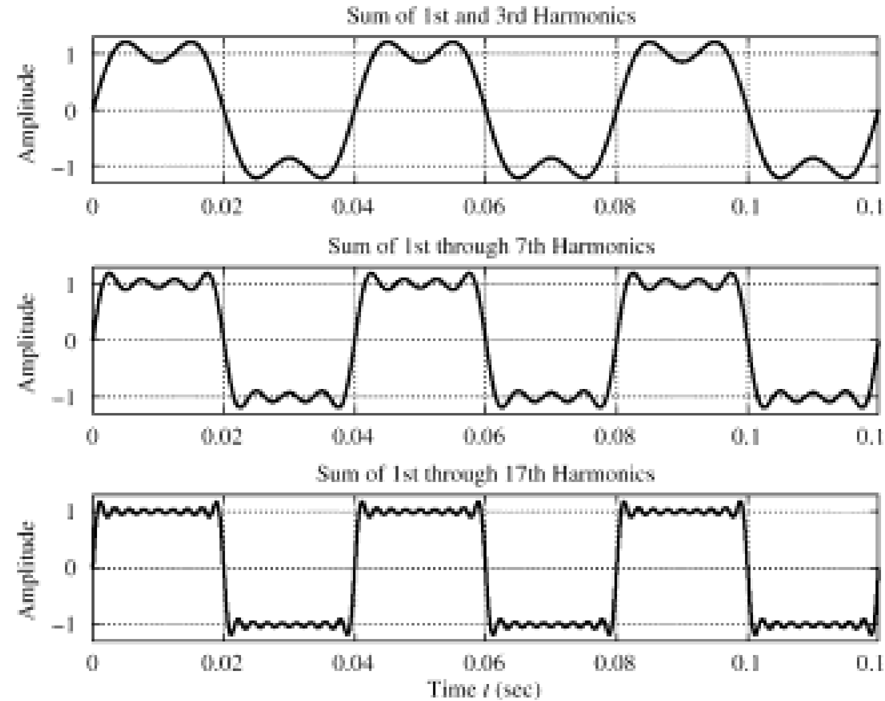


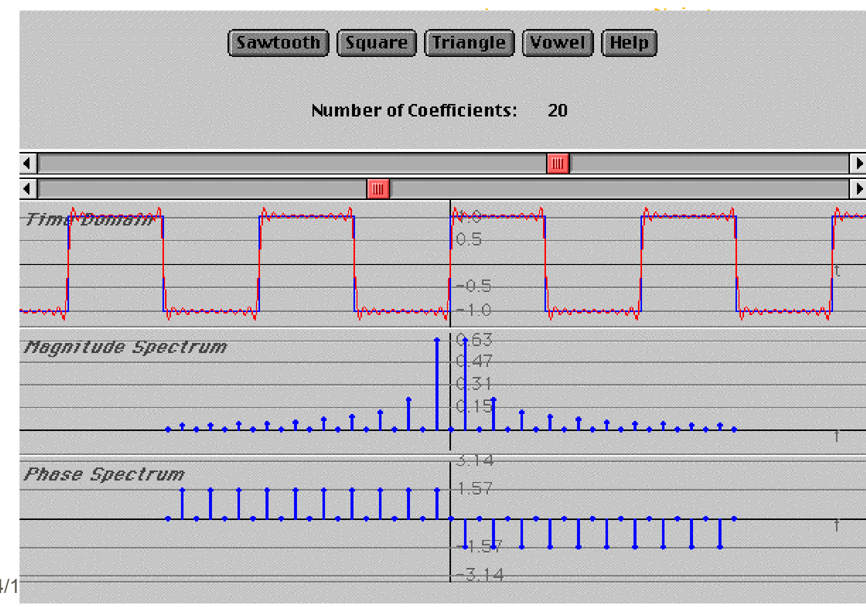
Figure 3.12 Spectrum of the square-wave signal whose Fourier series coefficients are given in (3.4.5) with $f_0 = 1/T_0 = 25$ Hz.

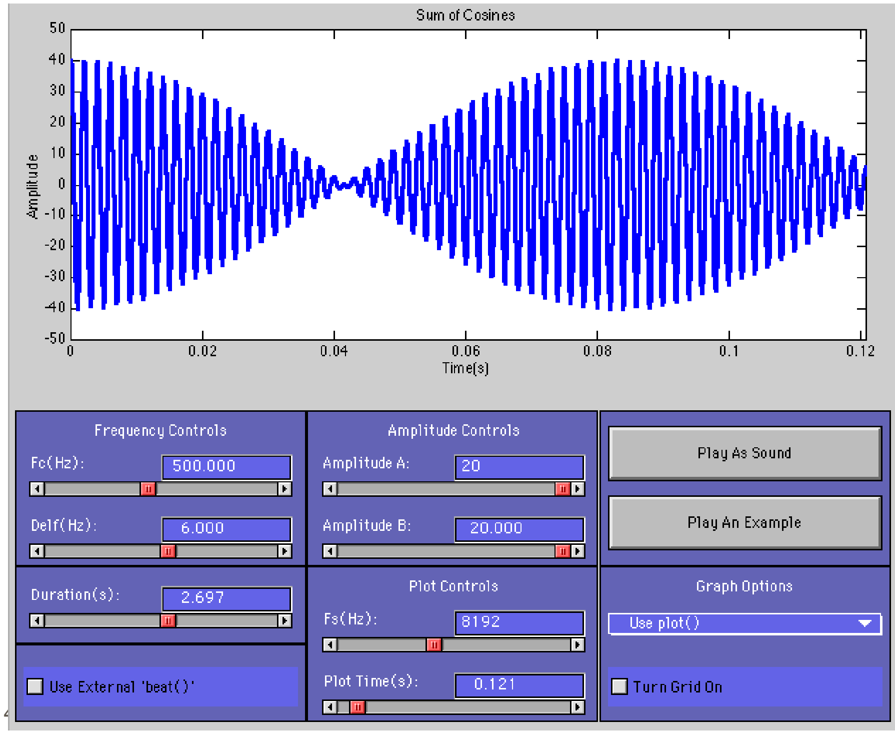


A Couple of DEMOS

- **Beat Control GUI**
  - **DSPFirst Toolbox: MATLAB**
    - **DSPFIRST/beatcon.m**
- **Fourier Series Java Applet**
    - **Interactive**
    - **http://users.ece.gatech.edu/~slabaugh/java/fourier/fourier.html**

Fourier Series Java Applet





Gibbs' Phenomenon

- Convergence at **DISCONTINUITY** of x(t)
  - There is always an **overshoot**
  - **9%** for the Square Wave case