GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL of ELECTRICAL and COMPUTER ENGINEERING

ECE 2025 Fall 2000
Lab #11: Design with Fourier Series

Date: 14–20 Nov. 2000

This is *the official* Lab #11 description. **It will be graded out of 150 points.**

The lab report for this lab will be **INFORMAL:** discuss your Fourier Series results from sections 4 and 5. Some of the questions require mathematical derivations to support the answer. Turn in the **Instructor Verification** sheet to your TA at the end of your lab.

The report will **due during the week of 28 Nov.–4 Dec. at the start of your lab.**

For the MONDAY Lab Sections ONLY: The Monday Lab sections will meet on 20 & 27 Nov. to do the warm-ups for Labs #11 and #12. The report for Lab #11 (this lab) will be due on Wednesday, 29-Nov. **(in Recitation).** The report for Lab #12 will be due on Monday, 4-Dec.

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1 Introduction & Objective

The goal of this laboratory project is to show that Fourier Series analysis is a powerful method for predicting the response of a LTI system when the input is a periodic signal. Since we will be doing Fourier Series for continuous-time signals, the formulas are integrals. As a result, we will use MATLAB's numerical integration capability to calculate the Fourier Series coefficients of the output and the input; this method was introduced in Lab #10. In this particular lab, we will use Fourier Series and the Fourier transform to analyze a frequency synthesis design problem in the frequency domain.

2 Background: Fourier Series Analysis and Synthesis

Recall the *analysis* integral and *synthesis* summation for the Fourier Series expansion of a periodic signal $x(t) = x(t + T_0)$. The Fourier synthesis equation for a periodic signal $x(t) = x(t + T_0)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}, \tag{1}$$

where $\omega_0 = 2\pi/T_0$ is the *fundamental* frequency. To determine the Fourier series coefficients from a periodic signal, we must evaluate the *analysis* integral for every integer value of k:

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-jk\omega_0 t} dt \tag{2}$$

where $T_0 = 2\pi/\omega_0$ is the *fundamental* period. If necessary, we can evaluate the analysis integral over any period; in (2) the choice $[-\frac{1}{2}T_0, \frac{1}{2}T_0]$ was a convenient one, but integrating over the interval $[0, T_0]$ would also give exactly the same answer.

The Fourier Series representation is extremely useful when studying the effects of an LTI filter, because the output signal is also periodic. The Fourier Series coefficients of the output signal $\{b_k\}$ are obtained by **multiplying** by the frequency response:

$$b_k = a_k\, H(j\omega_0 k) \tag{3}$$

where $H(j\omega_0 k)$ is the frequency response of the LTI system evaluated at the harmonics.

3 Warmup

In this project, we will use the Fourier Series coefficients to predict the response of a LTI system. Since you have already developed the capability to produce the Fourier Series numerically (in Lab #10), the primary point of the warm-up is to show that you can adapt your existing MATLAB functions to do a new example quickly. In particular, you will utilize your results from Lab #10 to write functions that will be able to

1. Evaluate the Fourier Series coefficients for the following periodic square-wave signal which is defined over one period to be:

$$x(t) = \begin{cases} A & \text{for } 0 \le t \le 0.2T_0 \\ 0 & \text{for } 0.2T_0 < t < T_0 \end{cases} \tag{4}$$

   The amplitude (A) and the length of the fundamental period (T_0) should be parameters that can be varied.

2. Synthesize approximations to $x(t)$ using a finite number of Fourier Series coefficients $\{a_k\}$. This is similar to the sum of sinusoids function written in Lab #3.

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{j2\pi kt/T_0}$$

   where $2N+1$ is the number of terms used to form the signal and T_0 is the period.
   Recall that in Lab #3, you were able to study the convergence of the Fourier Series as $N \to \infty$ by looking at synthesized waveforms.

If you are mathematically inclined, you might want to work out the Fourier Series integral for $x(t)$ in Eq. (4) by hand to show that the $\{a_k\}$ values are samples of a "sinc" function.

3.1 Square Wave Analysis

In this part of the warm-up, the objective is to make a plot of the spectrum versus frequency for the square wave defined above.

(a) The first step is to get the Fourier Series coefficients. Since you will be using `quad8()` to do the integration, write the auxiliary function that defines the Fourier Series integrand for the square wave defined in Eq. (4).

$$x(t)e^{-j2\pi kt/T_0}$$

Remember that it is necessary to introduce additional parameters, such as the ampliltude (A), the period (T_0), and the index k, when defining this auxiliary function.

(b) Write a MATLAB function that will evaluate the Fourier Series coefficients for the square wave over the range of indices $k = -N, \ldots, -1, 0, 1, 2, \ldots, N$. The function will contain a `for` loop to do all the coefficients from $k = -N$ to $k = +N$. It should return a vector containing $2N+1$ elements which are the $\{a_k\}$ coefficients. At this point the function should be general enough to work for any A and T_0.

(c) Use the Fourier Series function written in the previous part to evaluate the $\{a_k\}$ coefficients for $x(t)$ from Eq. (4) for the case where $A = 3\pi$, and $T_0 = 3.0$ secs. Find the $\{a_k\}$ coefficients for $N = 7$ and make a stem plot of the magnitude of the coefficients versus k.

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3.2 Fourier Synthesis (Optional)

In this part, you can synthesize approximations to $x(t)$ using a finite number of Fourier Series coefficients $\{a_k\}$. Let N denote the largest index used, so that $2N+1$ is the number of terms used to form the signal.

(a) Using your `vsynthesis` function (from Lab #3) for sinusoidal synthesis from a finite number of terms, generate $x_N(t)$ for $N = 7$ and $N = 15$.[1] For each of these synthesized signals, make a plot showing the synthesized signal and $x(t)$ on the same plot. Use a two-panel subplot to show the two cases in one figure window.

Reminder: You have to set up a time grid for the time interval, and you should use three periods of the signal from $t = -T_0$ to $t = 2T_0$. The spacing between grid points must be small enough so that the sampling rate (grid points per second) is at least *twice as high as the highest frequency component in your signal,* and even more oversampling will probably be needed to "see" convergence.

(b) Explain how you are getting convergence as N increases. Where does the approximation error seem to be largest?

3.3 Frequency Response of an Analog Filter

In the lab project, you will use a continuous-time LTI system for filtering. In this section of the warm-up, we will investigate the following frequency response:

$$H(j\omega) = \frac{j\omega}{(4 - \omega^2) + j\omega/3}$$

(a) In this part, you will have to make a plot of the magnitude and phase of $H(j\omega)$ versus frequency. In order to get values for the plot, you should evaluate the $H(j\omega)$ formula directly for a dense grid of frequencies. Use a range of frequencies that extends from -12 rad/s to $+12$ rad/s.[2] Plot $|H(j\omega)|$ versus ω. What kind of filter is $H(j\omega)$?

(b) Determine the peak value of the magnitude (frequency) response and the location of the peak. Evaluate the frequency response formula to verify that the peak value is correct.

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3.4 Filtering a Periodic Signal

In this part, you will have to "filter" the periodic input signal (the square wave from Section 3.1) through a continuous-time LTI system whose frequency response is given in Section 3.3. Since this is an analog system, we cannot do the filtering; instead, we calculate what the output signal would be by finding the Fourier Series of the output.

[1] In order to use the `vsynthesis()` function for Fourier synthesis, we must include all the complex $\{a_k\}$ coefficients for both the positive and negative k. Likewise the frequency vector, `fk` must contain both positive and negative harmonics.

[2] You can plot the frequency response versus frequency in hertz or radians/sec. Either way is acceptable, but make sure that you label the horizontal axis.

(a) Determine the Fourier Series coefficients of the output signal $y(t)$ when the input is the periodic square wave defined in (4). Use the frequency response $H(j\omega)$ and apply (3). Make a stem plot of the Fourier coefficients of $y(t)$ versus frequency, over the frequency range -12 rad/s to $+12$ rad/s. Recall that the Fourier coefficients are, in fact, the (magnitude) spectrum of $y(t)$.

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(b) Synthesize an approximation $y_N(t)$ by using $2N+1 = 7$ Fourier coefficients.

4 Lab Project: Analysis of an Analog Bandpass Filter

Generating a perfect sinusoid can be difficult if we want to get the wave shape exact. In fact, it is extremely rare to find hardware that can produce values that match $\cos(\omega t)$. On the other hand, it is relatively easy to generate a periodic signal with a switching circuit that is driven by a clock circuit. It turns out that we can filter the periodic signal to make a sinusoid. In this lab exercise, you will analyze a frequency synthesizer and design its parameters by using methods in the frequency domain.

The basic idea of the system is to use a bandpass filter to "clean up" the output of a "cheap" signal generator (Fig. 1). The signal generator only needs to be capable of making a periodic signal with the correct period.[3] Ideally, the bandpass filter will remove all the harmonics except one and make the output signal $y(t)$ a pure sinusoid.

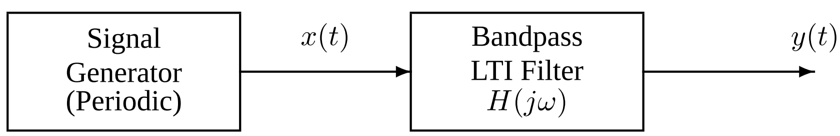

Figure 1: Block diagram representation of a Frequency Synthesizer.

The bandpass filter will be specified in the frequency domain by giving its frequency response (or Fourier transform) as:

$$H(j\omega) = \frac{j\omega G}{(a^2 - \omega^2) + j\omega b} \tag{5}$$

where the three parameters G, a and b can be manipulated to change the characteristics of the frequency response. There are three characteristics of interest: location of the passband (center frequency), width of the passband, and maximum height of the passband (probably at the center frequency). This particular frequency response corresponds to a resonant electric circuit that is relatively easy to build.

4.1 Fourier Analysis of the Input Waveform

Assume that the signal generator produces a periodic signal that is defined over one period by the following equation:

$$x(t) = \begin{cases} 1 & \text{for } 0 \le t \le \frac{1}{2}T_0 \\ 0 & \text{for } \frac{1}{2}T_0 < t < T_0 \end{cases} \tag{6}$$

The periodic signal $x(t)$ is a square wave with a period of T_0. In order to do the system design in the frequency domain, we need to get the Fourier Series of the square wave, and plot its spectrum.

[3] For example, the signal generator might produce a triangle wave or a square wave or a sawtooth wave for $x(t)$.

(a) As in the warm-up, you should use `quad8()` to do the Fourier integration. Therefore, you must write an auxiliary function that defines the Fourier Series integrand for the square wave defined in Eq. (6). Remember that it is necessary to introduce other parameters, such as T_0 and k, when defining this auxiliary function.

(b) Write a MATLAB function that will evaluate the Fourier Series coefficients for the square wave over the range of indices $k = -N, \ldots, -1, 0, 1, 2, \ldots, N$. The function will contain a `for` loop to do all the coefficients from $k = -N$ to $k = +N$. It should return a vector containing $2N+1$ complex elements which are the $\{a_k\}$ coefficients.

(c) Use the Fourier Series function written in the previous part to evaluate the Fourier Series coefficients for $x(t)$ from Eq. (6) for the case where $T_0 = 10^{-7}$ secs (100 nanoseconds). Find the $\{a_k\}$ coefficients for $N = 7$ and make a stem plot of the magnitude of the Fourier coefficients versus k.

(d) Finally, as a confirmation that you can analyze this waveform, you should also derive the mathematical expression for the Fourier Series coefficients. Consult the Fourier Series notes (Chapter 3 supplement) for a similar example that grinds out the integrals. Give your formula for the $\{a_k\}$ coefficients in polar form with a magnitude and phase.

4.2 Frequency Response of the Bandpass Filter

Now we must learn a little more about the bandpass filter parameters: G, a and b which are part of the frequency response defined in (5).

(a) Write a MATLAB function that will plot the magnitude and phase of $H(j\omega)$ defined in (5) over the range $-1.5\pi(10^8) \leq \omega \leq 1.5\pi(10^8)$. The function should have at least four input arguments: G, a and b, as well as a frequency vector for ω. Write this code from scratch using `plot()` or `fplot()`; don't use MATLAB's built-in analog frequency response function.[4] Test your function by making the plot for the following values of G, a and b

$$G = 32 \times 10^7, \qquad a = 5 \times 10^7, \qquad b = 8 \times 10^7$$

4.3 Find the Output Signal and Spectrum

The Fourier coefficients of the output signal are $b_k = a_k H(jk\omega_0)$, because the theory of the frequency response tells us how to determine the exact output of the lowpass filter by tracking each sinusoidal component through the filter. Using our $2N+1$ term approximation for the input, the approximate output is

$$y_N(t) = \sum_{k=-N}^{N} b_k e^{jk\omega_0 t} = \sum_{k=-N}^{N} a_k H(jk\omega_0) e^{jk\omega_0 t} \tag{7}$$

where the a_k are the Fourier coefficients of $x(t)$.

(a) *Frequency Domain:* Make a three-panel plot showing the spectrum of $x_N(t)$ in the top for $N = 7$; the magnitude of $H(j\omega)$ in the middle; and the **spectrum** of $y_N(t)$ in the bottom plot (for $N = 7$ also). For $H(j\omega)$ use the specific values for G, a and b given in the previous section.
Use the same horizontal frequency scale for all three plots so that they line up. Note the peak location of the frequency response, and explain how it is related to the parameter a. Explain how the output spectrum is produced from the input spectrum.

[4]You cannot use `freqz()` because this is not a digital filter; MATLAB has a function called `freqs()`, but you shouldn't use this function in this lab.

(b) *Time Domain:* Next, you should make a plot of the output signal in the time domain for $N = 7$, i.e., plot $y_7(t)$ versus t over the range $-T_0 \le t \le 2T_0$.

This requires that you evaluate the b_k Fourier coefficients numerically and use `vsynthesis` to create $y_N(t)$. In this approach, use the Fourier coefficients a_k that were evaluated numerically, and then evaluate the frequency response $H(j\omega)$ at the appropriate frequencies. Recall that the Fourier synthesis of the output is given by Eq. (7).
Is the output signal a sinusoid? How close is $y(t)$ to being a sinusoid? What is its frequency?

5 Lab Project: Design of a Frequency Synthesizer

Now we turn our attention to the design of the BPF to produce the desired output sinusoid with control over the amount of distortion.

5.1 Design in the Frequency Domain

The objective of the design is to produce an output $y(t)$ that is a pure sinusoid at one given frequency. The approach is to use the system in Fig. 1 and design the bandpass filter to pass the first harmonic of $x(t)$, and reject all others. Thus we need to position the passband of the BPF correctly.

(a) For the square-wave input signal we will set $T_0 = 1 \times 10^{-7}$ sec for the rest of the lab project. Determine the fundamental frequency of the input signal $x(t)$. In addition, for the Fourier Series of the square signal, record the value of first three harmonics, i.e., the magnitude and phase of a_1, a_2, and a_3.

(b) In this part, we will design the frequency response of $H(j\omega)$ in (5). Let the values of G and a be unknown parameters, but constrain the value of b to such that $b = a/2$. Use your knowledge of the input Fourier series and $H(j\omega)$ to write a mathematical formula for the first-harmonic component of the output (in terms of G and a). Hint: Use the frequency response $H(j\omega)$ to find its value at the fundamental frequency in terms of G and a.

(c) Pick the value of a so that the amplitude of the output sinusoid will be maximized. This choice will position the peak of the frequency response.

(d) The output will not be an exact sinusoid, but the dominant part due to the first harmonic will be a sinusoid. Ignoring the distortion in the output, determine the frequency of the output signal.

(e) Determine the value of G so that the output signal (as a sinusoid) has an amplitude of 100.

(f) Determine the DC value of the output signal.

5.2 Evaluate the Output Distortion

The frequency synthesizer will exhibit some distortion because the BPF is not perfect, and we have imposed the constraint $b = a/2$. We need to measure the distortion, so we will use the worst-case error between the perfect sinusoid and the actual output.

(a) First of all, determine the distortion (error) for the parameters found in the previous section. This should be carried out by taking the difference between the desired sinusoidal output and the actual synthesized output which can be produced via a Fourier Series synthesis for $N = 7$. Give your answer as a relative error, i.e., worst-case error divided by the desired amplitude of the output sinusoid.

(b) Make a plot of the *relative output error* signal over three periods. Mark the location(s) and value of the worst-case error on the plot.

5.3 Generalize the Design to Meet Distortion Specs

Now we can complete a general design of the frequency synthesizer for any specification on the amplitude of the output sinusoid and its distortion. Suppose that our design objective is to generate an output sinusoid whose specifications are:

$$\text{Amplitude} = 100 \qquad \text{Frequency} = 10^7 \text{ Hz} \qquad \text{Distortion} < 1\%$$

(a) In this case, we must let the parameter b in the frequency response be a third unknown. However, we already know that we can find some of the parameters independently. For example, since we want the output frequency to be $2\pi \times 10^7$ rad/s, we can determine the value of a.

(b) When you made the plot of $y_7(t)$ in the previous section, you should have observed a "distortion" in the time-domain signal. In other words, the output was not a perfect sinusoid. The distortion in the output shows up as extra bumps in the waveform, and is due to all the non-first-harmonic terms in the output Fourier Series, but it is mostly due to the terms for $k = \pm 3$ (because b_2 is zero).

$$y(t) = b_0 + \left(b_1 e^{j\omega_0 t} + b_{-1} e^{-j\omega_0 t}\right) + \sum_{k=2}^{\infty} \left(b_k e^{jk\omega_0 t} + b_{-k} e^{-jk\omega_0 t}\right)$$

Therefore, the output can be well approximated by considering the signal $y_3(t)$ that contains only two sinusoidal terms plus DC. You should derive the mathematical formula for $y_3(t)$ in terms of the unknown parameters G, a and b. When you assess the quality of the frequency synthesis (below), you only need the mathematical formula for the *magnitude* of the first two non-zero sinusoidal terms in $y_3(t)$ (the phase is less important).

(c) If we assume that all of the error is caused by the third harmonic, then we can write an expression for the error in terms of G and b (a was already determined in part(a)). Likewise, the desired amplitude specification leads to a second equation for G and b to produce the correct amplitude. These two equations (in two unknowns) can be solved for the remaining parameters of the BPF.

(d) Solve the two equations from the previous part for the specifications above.

(e) For the parameters found in the previous part, use MATLAB to find the Fourier Series coefficients of the output signal and make a plot of the spectrum versus frequency.

(f) Next, make a plot of the output waveform over three periods by doing a Fourier Series synthesis with $N = 7$. In addition, make a plot of the relative error signal over three periods. Put these two plots into a single figure window with `subplot()`. Mark the worst-case error on the plot. Evaluate the Fourier coefficient b_3 for the third harmonic and write this value on the plot. How should the size of the error be related to b_3?

(g) Does the output signal look like a pure sinusoid with amplitude 100? Explain why or why not. Explain the validity of using one Fourier coefficient for the design. In other words, is your error signal exactly the size predicted by the design formula? Make a recommendation on whether or not you should include the *fifth* harmonic in the design. Do you think it would be easy to extend the design procedure to include the fifth harmonic? If you think you can derive these more accurate design equations, then describe how to do it (or go ahead and carry out the math).

Lab #11
ECE-2025 Fall-2000
Instructor Verification

Name: ________________________________ Date of Lab: __________

Part 3.1 Illustrate the Fourier analysis for a square wave. Find the $\{a_k\}$ coefficients numerically. Plot the spectrum of the square wave versus frequency.[5]

Verified:____________________ Date/Time:__________

Part 3.3 Plot the magnitude and phase of the frequency response of a continuous-time filter, $H(j\omega)$. Draw a sketch of the magnitude in the space below. Indicate the peak location and peak value.

Verified:____________________ Date/Time:__________

Part 3.4 Find the $\{b_k\}$ Fourier Series coefficients of the output signal. Plot the spectrum versus frequency of the output signal.

Verified:____________________ Date/Time:__________

[5] It might be natural to plot a_k versus k, but when you show the spectrum hte horizontal axis must be frequency (in Hz or rad/s).