ECE 2025 HW#4 Solutions Fall 2001

4.1) a) $x(t) = \left[10 + \frac{5}{2} e^{-j\frac{\pi}{2}} e^{j30{,}000\pi t} + \frac{5}{2} e^{j\frac{\pi}{2}} e^{-j30{,}000\pi t}\right]$

$$\times \frac{1}{2}\left[e^{j\frac{\pi}{2}} e^{j100{,}000\pi t} + e^{-j\frac{\pi}{2}} e^{-j100{,}000t}\right]$$

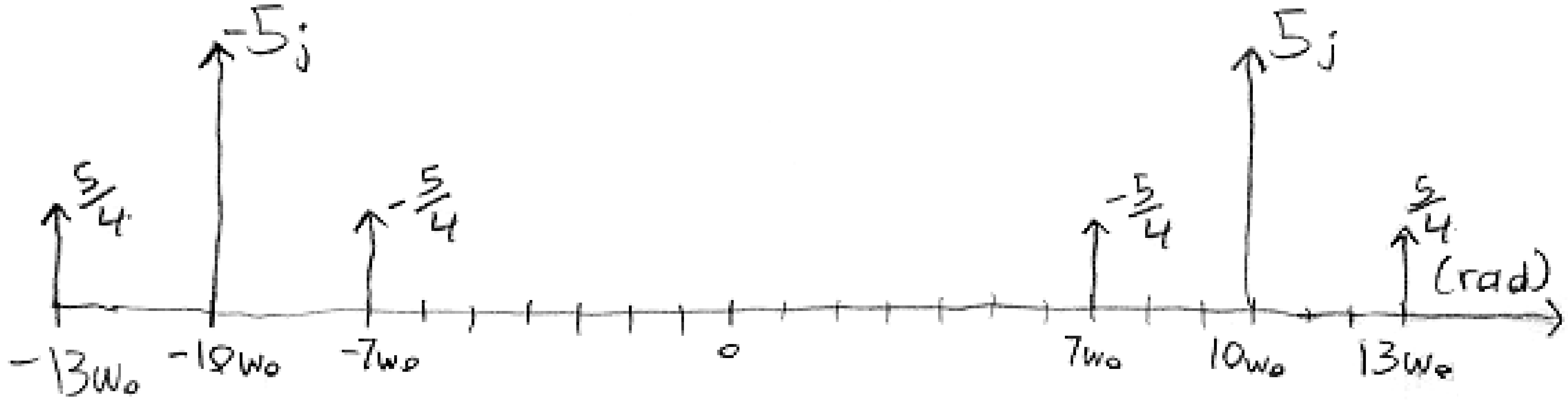
$$= \boxed{\begin{aligned} &5e^{j\frac{\pi}{2}} e^{j100{,}000\pi t} + 5e^{-j\frac{\pi}{2}} e^{-j100{,}000\pi t} \\ &+ \frac{5}{4} e^{j130{,}000\pi t} + \frac{5}{4} e^{-j130{,}000\pi t} \\ &+ \frac{5}{4} e^{j\pi} e^{j70{,}000\pi t} + \frac{5}{4} e^{-j\pi} e^{-j70{,}000t} \end{aligned}}$$

b) The GCD of $70{,}000\pi$, $100{,}000\pi$, and $130{,}000\pi$ is $10{,}000\pi$, hence $\boxed{\omega_0 = 10{,}000\pi}$.

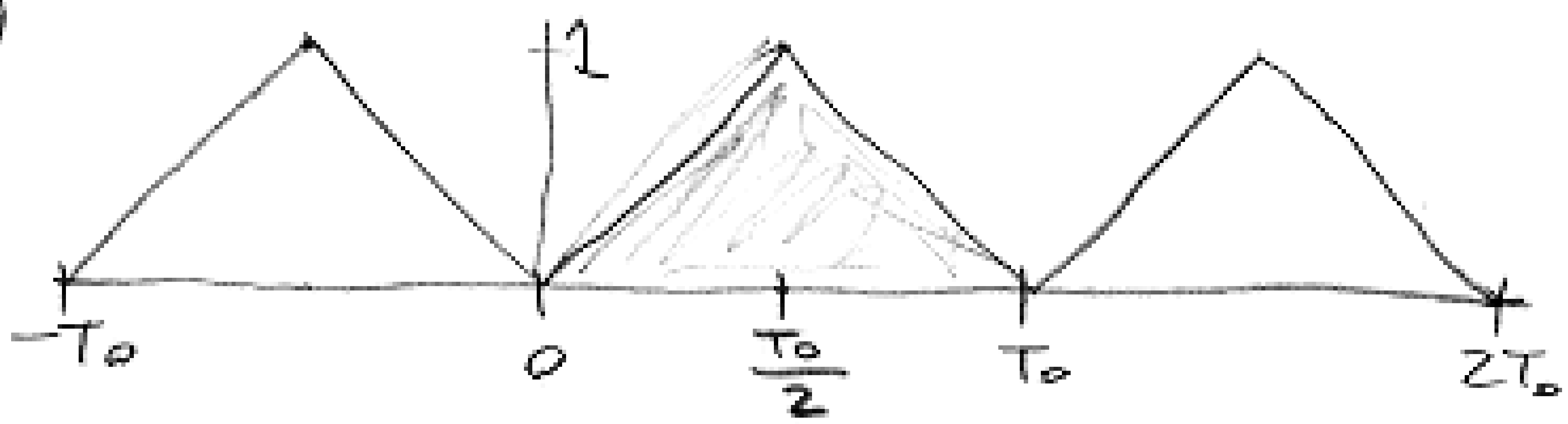
c) There is no zero frequency component, so "DC value" $= \boxed{0}$

d) $a_7 = \frac{5}{4} e^{j\pi} = -\frac{5}{4}$, $a_{10} = 5e^{j\frac{\pi}{2}} = 5j$, $a_{13} = \frac{5}{4}$

$a_{-7} = \frac{5}{4} e^{-j\pi} = -\frac{5}{4}$, $a_{-10} = 5e^{-j\frac{\pi}{2}} = -5j$, $a_{-13} = \frac{5}{4}$



4.2) a)



b) $a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0} \times \{\text{Area of shaded triangle}\}$

$$= \frac{1}{T_0} \times \frac{T_0}{2} = \boxed{\frac{1}{2}}$$

4.3) a) $a_k = \frac{1}{T_0}\left\{\int_{-\frac{T_0}{2}}^{0}\left(-\frac{2t}{T_0}\right)e^{-jk\frac{2\pi}{T_0}t}\,dt + \int_0^{\frac{T_0}{2}}\left(\frac{2t}{T_0}\right)e^{-jk\frac{2\pi}{T_0}t}\,dt\right\}$

Use integration by parts. <u>Let $b = -jk\frac{2\pi}{T_0}$</u>

$$\int \underbrace{t}_{=u}\underbrace{e^{bt}\,dt}_{=dv} = uv - \int v\,du = \frac{t}{b}e^{bt} - \int \frac{1}{b}e^{bt}\,dt$$

$du = dt \quad v = \frac{1}{b}e^{bt}$

$$= \frac{t}{b}e^{bt} - \frac{1}{b^2}e^{bt}$$

$$= e^{bt}\left[\frac{t}{b} - \frac{1}{b^2}\right]$$

$$a_k = \frac{1}{T_0}\left\{\frac{-2}{T_0}\left[e^{bt}\left(\frac{t}{b} - \frac{1}{b^2}\right)\right]\Bigg|_{t=\frac{-T_0}{2}}^{t=0} + \frac{2}{T_0}\left[e^{bt}\left(\frac{t}{b} - \frac{1}{b^2}\right)\right]\Bigg|_{t=0}^{t=\frac{T_0}{2}}\right\}$$

$$= -\frac{2}{T_0^2}\left[-\frac{1}{b^2}\right] + \frac{2}{T_0^2}\left[e^{-b\frac{T_0}{2}}\left(\frac{-T_0/2}{b} - \frac{1}{b^2}\right)\right]$$

$$+ \frac{2}{T_0^2}\left[e^{b\frac{T_0}{2}}\left(\frac{T_0/2}{b} - \frac{1}{b^2}\right)\right] - \frac{2}{T_0^2}\left[-\frac{1}{b^2}\right]\Bigg\}$$

$\rightarrow$

4.3 a cont'

$$= \frac{2}{T_0 b^2}\left[2 - e^{-b\frac{T_0}{2}} - e^{b\frac{T_0}{2}}\right] + \frac{2}{T_0 b}\left[-\frac{T_0}{2}e^{-b\frac{T_0}{2}} + \frac{T_0}{2}e^{b\frac{T_0}{2}}\right]$$

Insert $b = -jk\frac{2\pi}{T_0}$

$$= \frac{2\,\cancel{T_0^2}}{-\cancel{T_0^2}k^2 4\pi^2}\left[2 - e^{+jk\pi} - e^{-jk\pi}\right] + \frac{2}{-\cancel{T_0}jk\frac{2\pi}{\cancel{T_0}}}\left[-\frac{T_0}{2}e^{jk\pi} + \frac{T_0}{2}e^{-jk\pi}\right]$$

(pulled out a 2)

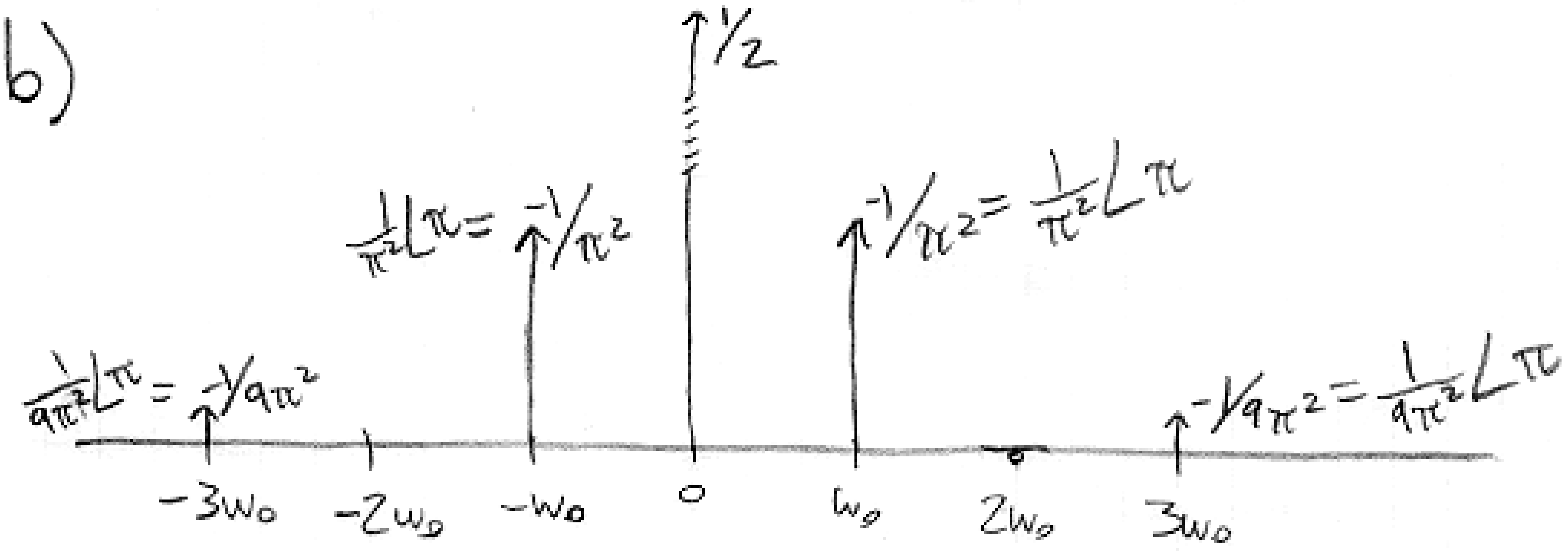
$$= \frac{1}{k^2\pi^2}\left[\cos(k\pi) - 1\right] + \cancel{\frac{1}{k\pi}\left[\sin(k\pi)\right]} = 0$$

(for integer k)

$$= \frac{1}{k^2\pi^2} \times \begin{cases} -2 & \text{for } k \text{ odd} \\ 0 & \text{for } k \text{ even} \end{cases}$$

$$\boxed{a_k = \begin{cases} -\frac{2}{k^2\pi^2} & \text{for } k \text{ odd} \\ 0 & \text{otherwise} \end{cases} \quad \text{for } k \neq 0}$$

b)



$$a_1 = -\frac{1}{\pi^2} \approx -0.1$$

$$a_3 = -\frac{1}{9\pi^2} \approx 0.01$$

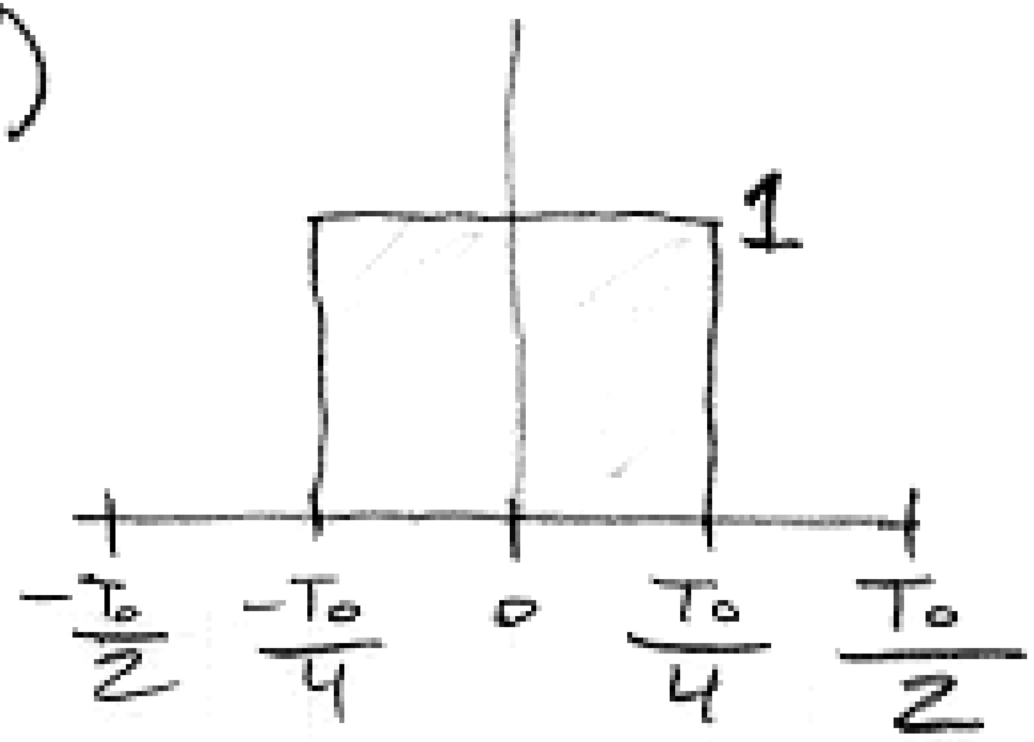
4.4) a) $y(t) = Ax(t) = A\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$= \sum_{k=-\infty}^{\infty} \underbrace{A a_k}_{=b_k} e^{jk\omega_0 t}$$

b) $y(t) = x(t-t_d) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t-t_d)}$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\boxed{a_k e^{-jk\omega_0 t_d}}}_{=b_k} e^{jk\omega_0 t}$$

4.5) a)



$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,dt$$
$$= \frac{1}{T_0} \times \{\text{Area of square}\}$$
$$= \frac{1}{T_0}\frac{T_0}{2} = \boxed{\frac{1}{2}}$$

For $k \neq 0$: $a_k = \frac{1}{T_0}\int_{-\frac{T_0}{4}}^{\frac{T_0}{4}} e^{-jk\omega_0 t}\,dt$ (using $\omega_0 = \frac{2\pi}{T_0}$)

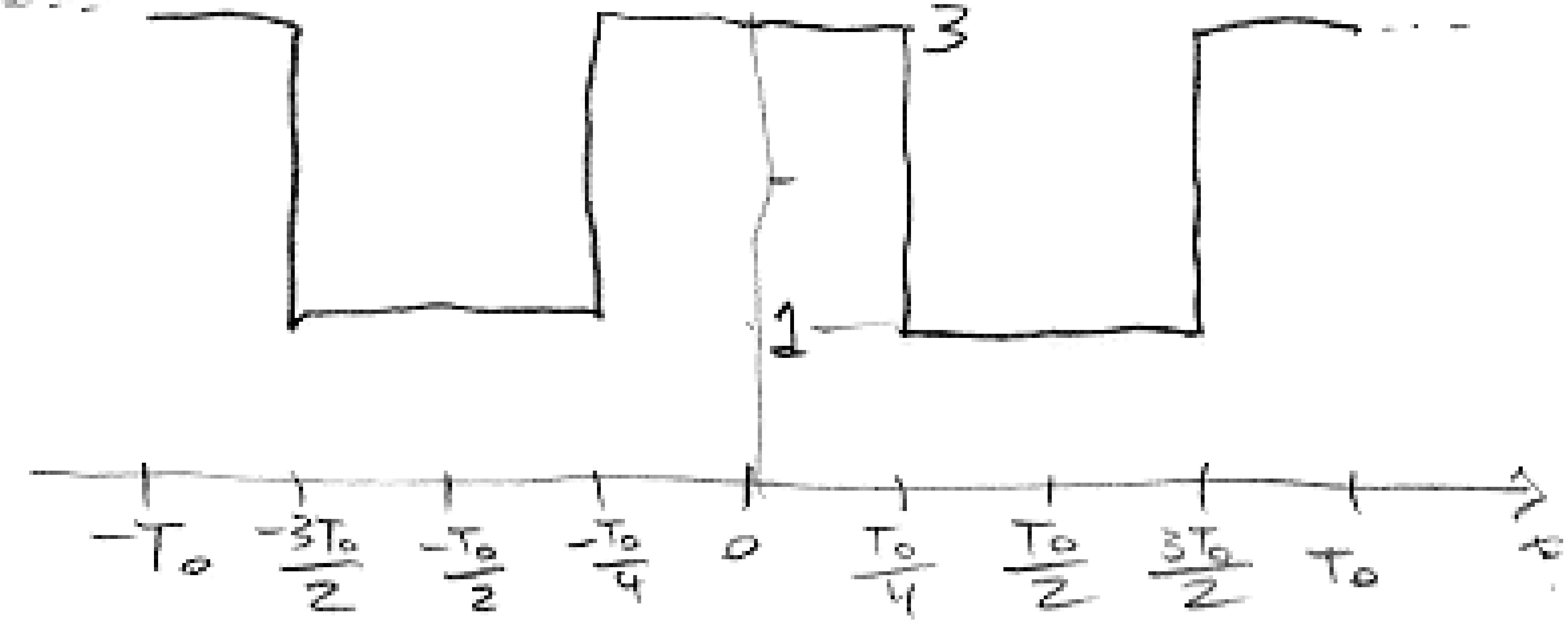
$$= \frac{1}{\cancel{T_0}}\left[-\frac{1}{jk\frac{2\pi}{\cancel{T_0}}} e^{-jk\frac{2\pi}{T_0}t}\right]_{t=-\frac{T_0}{4}}^{t=\frac{T_0}{4}}$$

$$= \frac{1}{k\pi}\left[\underbrace{\frac{1}{2j}\left(-e^{-jk\frac{2\pi}{T_0}\frac{T_0}{4}} + e^{jk\frac{2\pi}{T_0}\frac{T_0}{4}}\right)}_{\text{(inverse Eulers formula)}}\right]$$

$$= \frac{1}{k\pi}\sin\left(\frac{\pi}{2}k\right)$$

$$= \boxed{\begin{cases} \frac{1}{k\pi} & \text{for } k = \ldots, -7, -3, 1, 5, 9 \ldots \\ -\frac{1}{k\pi} & \text{for } k = \ldots, -5, -1, 3, 7, 11, \ldots \\ 0 & \text{otherwise} \end{cases}}$$

4.5) b)



By 4.4a, $b_0 = 2\cdot\underbrace{\frac{1}{2}}_{(a_0)} + 1 = 1+1 = \boxed{2}$

and

$$b_k = 2a_k = \boxed{\begin{cases} \frac{2}{k\pi} & \text{for } k = \ldots, -7, -3, 1, 5, 9 \ldots \\ -\frac{2}{k\pi} & \text{for } k = \ldots, -5, -1, 3, 7, 11, \ldots \\ 0 & \text{otherwise} \end{cases}}$$

c) $c_0 = a_0 = \boxed{\frac{1}{2}}$ (by 4.4b)

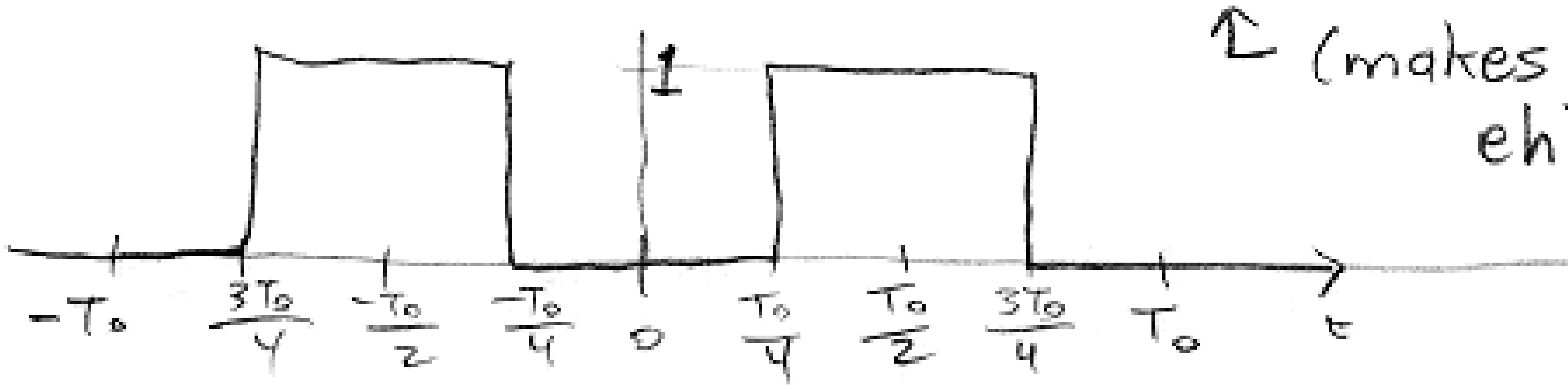
$$c_k = a_k e^{-jk\omega_0 \frac{T_0}{2}} = a_k e^{-jk\frac{2\pi}{\cancel{T_0}}\frac{\cancel{T_0}}{2}} = a_k e^{-jk\pi}$$

(note only need to think about k odd)

$$= -a_k$$

$$= \boxed{\begin{cases} \frac{-1}{k\pi} & \text{for } k = \ldots, -7, -3, 1, 5, 9, \ldots \\ \frac{1}{k\pi} & \text{for } k = \ldots, -5, -1, 3, 7, 11, \ldots \\ 0 & \text{otherwise} \end{cases}}$$

(makes sense, eh?)

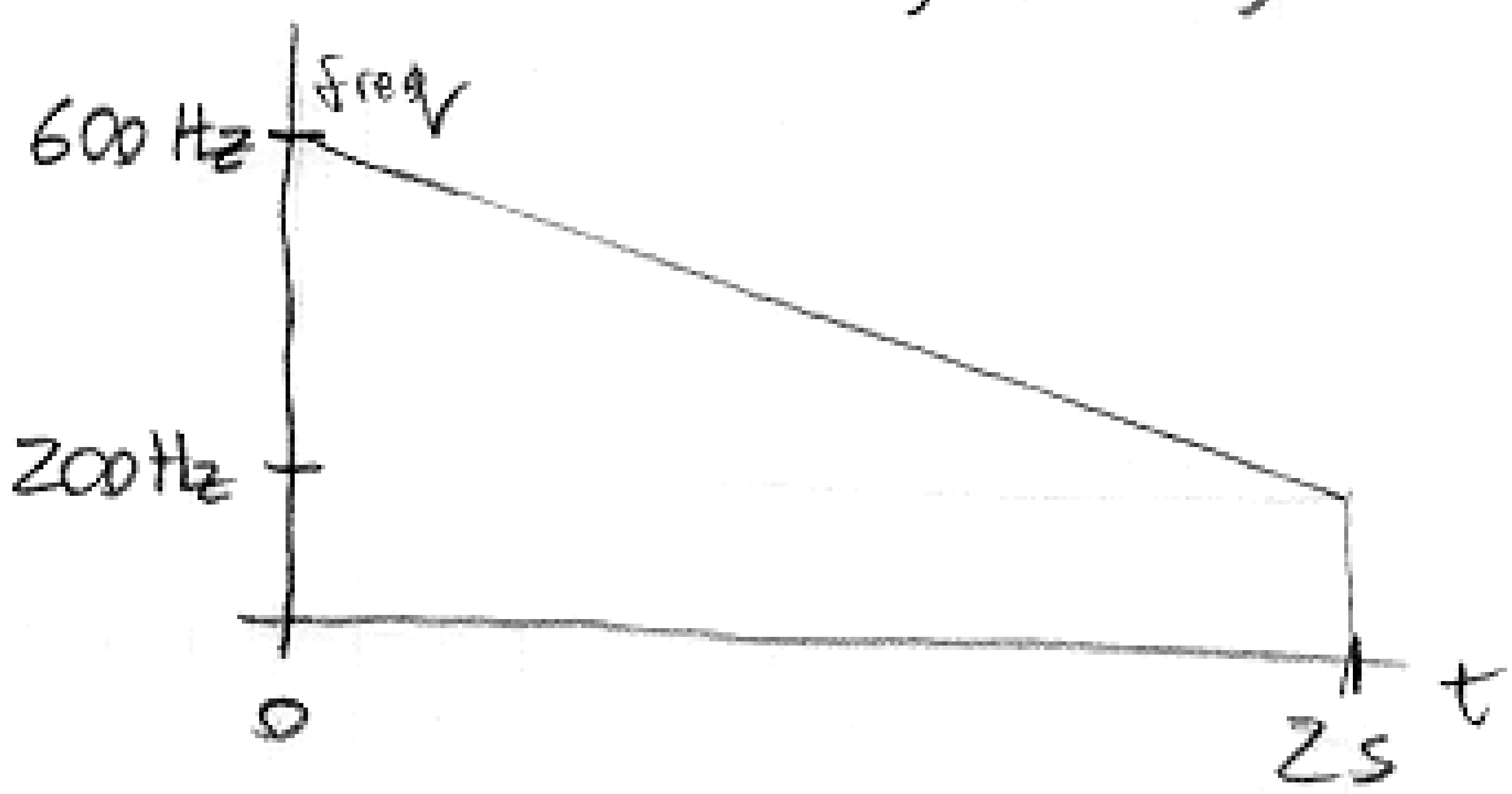


4.6) a) $x(t) = \cos[\underbrace{2\pi(-100t^2+600t+66)}_{=\Psi(t)}]$

$$\omega_i(t) = \frac{d}{dt}\Psi(t) = \boxed{2\pi(-200t+600)} \quad \text{(rad)}$$

b) $f_i(t) = \frac{\omega_i(t)}{2\pi} = -200t + 600 \quad (Hz)$

$f_i(0) = 600\,Hz$, $f_i(2) = -200\cdot 2 + 600 = 200\,Hz$



c) In general, $f_i(t) = 2\alpha t + B$

We need $f_i(0) = 2\alpha\cdot 0 + B = 5000\,Hz$, so $B = 5000$

and $f_i(2) = 2\alpha\cdot 2 + B = 4\alpha + 5000 = 1000$

$4\alpha = -4000$

so $\alpha = -1000$

hence $x(t) = \boxed{\cos[2\pi(-1000t^2+5000t)]}$

d)