ECE-2025 HW #7 Fall-2001

From Problem Set #6 Solution Sp. 99

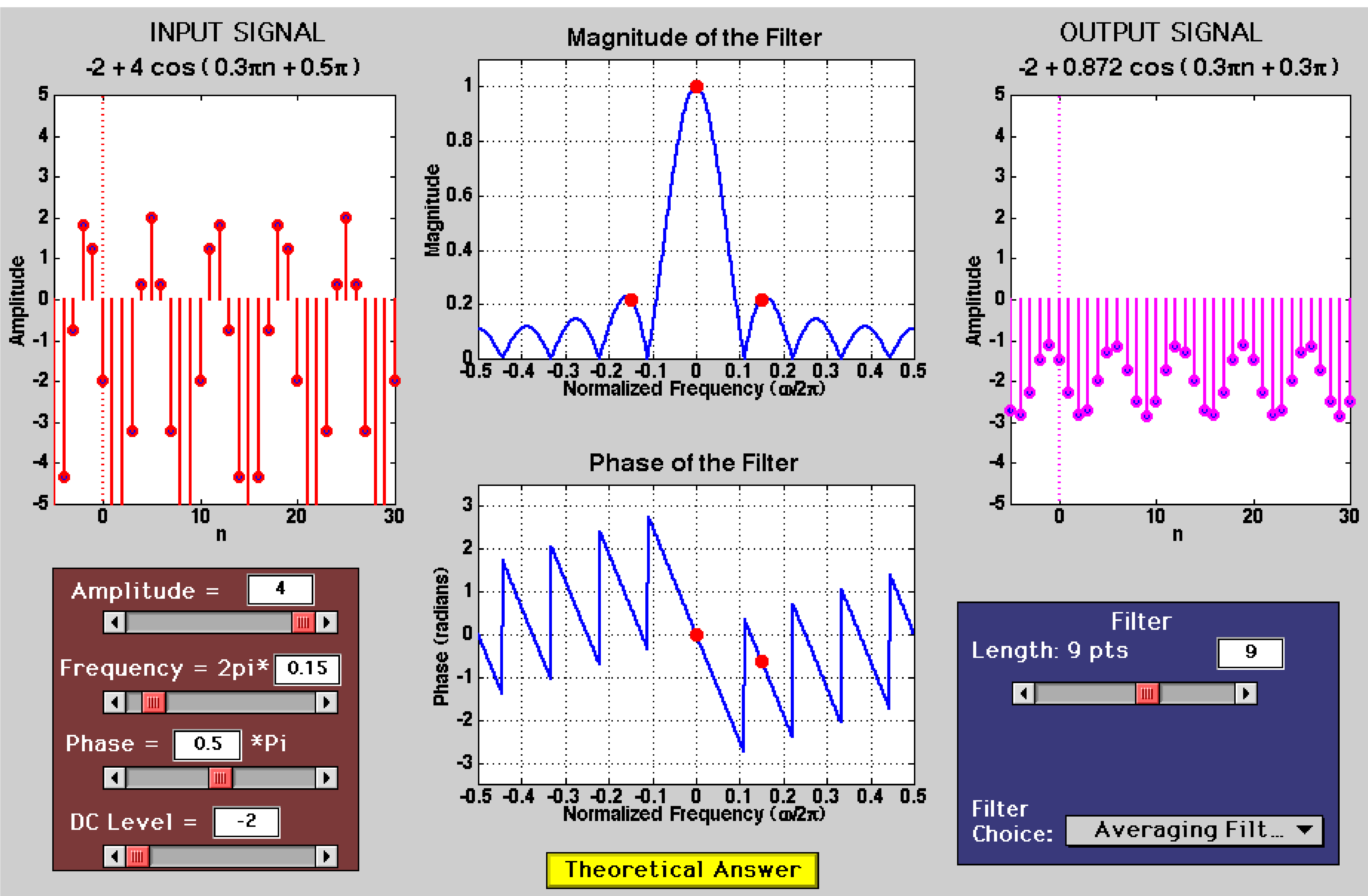
6.1 a) From left to right, the spectral lines correspond to the following time signals:

$$2e^{-j\pi/2}e^{-j0.3\pi n}, \quad 3e^{j\pi}, \quad \text{and}$$

$$2e^{j\pi/2}e^{j0.3\pi n}$$

X[n] is the sum of these, or

$$x[n] = 4\cos(0.3\pi n + \pi/2) - 3$$



6.1

(b) The screen shot is included. Since the present version of the GUI only permits a DC value of -2, the input was set equal to

$$x[n] = -2 + 4\cos(0.3\pi n + 0.5\pi)$$

The calculated output is

$$y[n] = -2 + 0.872\cos(0.3\pi n + 0.3\pi)$$

This would be -3 if the DC component of $x[n]$ were equal to -3.

(c) Use the frequency response. Evaluate at $\hat{\omega} = 0$ and at $\hat{\omega} = 0.3\pi$.

$$\mathcal{H}(\hat{\omega}) = \frac{1}{9}\sum_{k=0}^{8} e^{-j\hat{\omega}k} = e^{-j4\hat{\omega}} \frac{\sin 9\hat{\omega}/2}{9\sin \hat{\omega}/2}$$

At $\hat{\omega} = 0$ $\quad \mathcal{H}(\hat{\omega}) = \frac{1}{9}\sum_{k=0}^{8} e^{-j0} = \frac{1}{9}(9) = 1$

At $\hat{\omega} = 0.3\pi$

$$\mathcal{H}(\hat{\omega}) = e^{-j4(0.3\pi)} \frac{\sin(\frac{9}{2}(0.3\pi))}{9\sin(\frac{1}{2}(0.3\pi))}$$

This term is negative

$$= e^{-j1.2\pi} \frac{\sin(1.35\pi)}{9\sin(0.15\pi)} = 0.218\, e^{-j0.2\pi}$$

The output is:

$$y[n] = \underbrace{(-3)\mathcal{H}(0)}_{-3} + \underbrace{4|\mathcal{H}(0.3\pi)|}_{0.872}\cos(0.3\pi n \underbrace{+0.5\pi + \angle\mathcal{H}(0.3\pi)}_{0.3\pi})$$

Prob 7.2

(a) All three are TRUE. The system is Linear, Time-Inv & Causal

Linearity ?

If $x_1[n] \to y_1[n]$, then $y_1[n] = x_1[n] + 3x_1[n] + x_1[n]$

If $x_2[n] \to y_2[n]$, then $y_2[n] = x_2[n] + 3x_2[n] + x_2[n]$

Define $v[n] = \alpha_1 x_1[n] + \alpha_2 x_2[n]$

Now, find the output when $v[n]$ is the input. Call this output $w[n]$.

$$w[n] = v[n] + 3v[n-1] + v[n-2]$$

$$w[n] = (\alpha_1 x_1[n] + \alpha_2 x_2[n]) + 3(\alpha_1 x_1[n-1] + \alpha_2 x_2[n-1]) + (\alpha_1 x_1[n-2] + \alpha_2 x_2[n-2])$$

$$w[n] = \underbrace{\alpha_1 x_1[n] + 3\alpha_1 x_1[n-1] + \alpha_1 x_1[n-2]}_{\alpha_1 y_1[n]} + \underbrace{\alpha_2 x_2[n] + 3\alpha_2 x_2[n-1] + \alpha_2 x_2[n-2]}_{\alpha_2 y_2[n]}$$

Since $w[n] = \alpha_1 y_1[n] + \alpha_2 y_2[n]$, the system is LINEAR

Time-Invariant ?

Assume $x[n] \to y[n]$. Define $v[n] = x[n-n_0]$

Find the output when $v[n]$ is the input. Call this output $r[n]$.

$$r[n] = v[n] + 3v[n-1] + v[n-2]$$

$$= x[n-n_0] + 3x[n-1-n_0] + x[n-2-n_0]$$

$$= x[(n-n_0)] + 3x[(n-n_0)-1] + x[(n-n_0)-2] = y[n-n_0]$$

Since $r[n] = y[n-n_0]$, the system is Time-Invariant.

Causal ?

The system is causal because $y[n] = x[n] + 3x[n-1] + x[n-2]$ uses $x[n]$, $x[n-1]$, and $x[n-2]$ to form $y[n]$.

present

past

(b) The filter coefficients are $\{b_k\} = \{1, 3, 1\}$

$$\mathcal{H}(\hat{\omega}) = 1 + 3e^{-j\hat{\omega}} + e^{-j\hat{\omega}} \leftarrow \sum_{k=0}^{M} b_k e^{-jk\hat{\omega}}$$

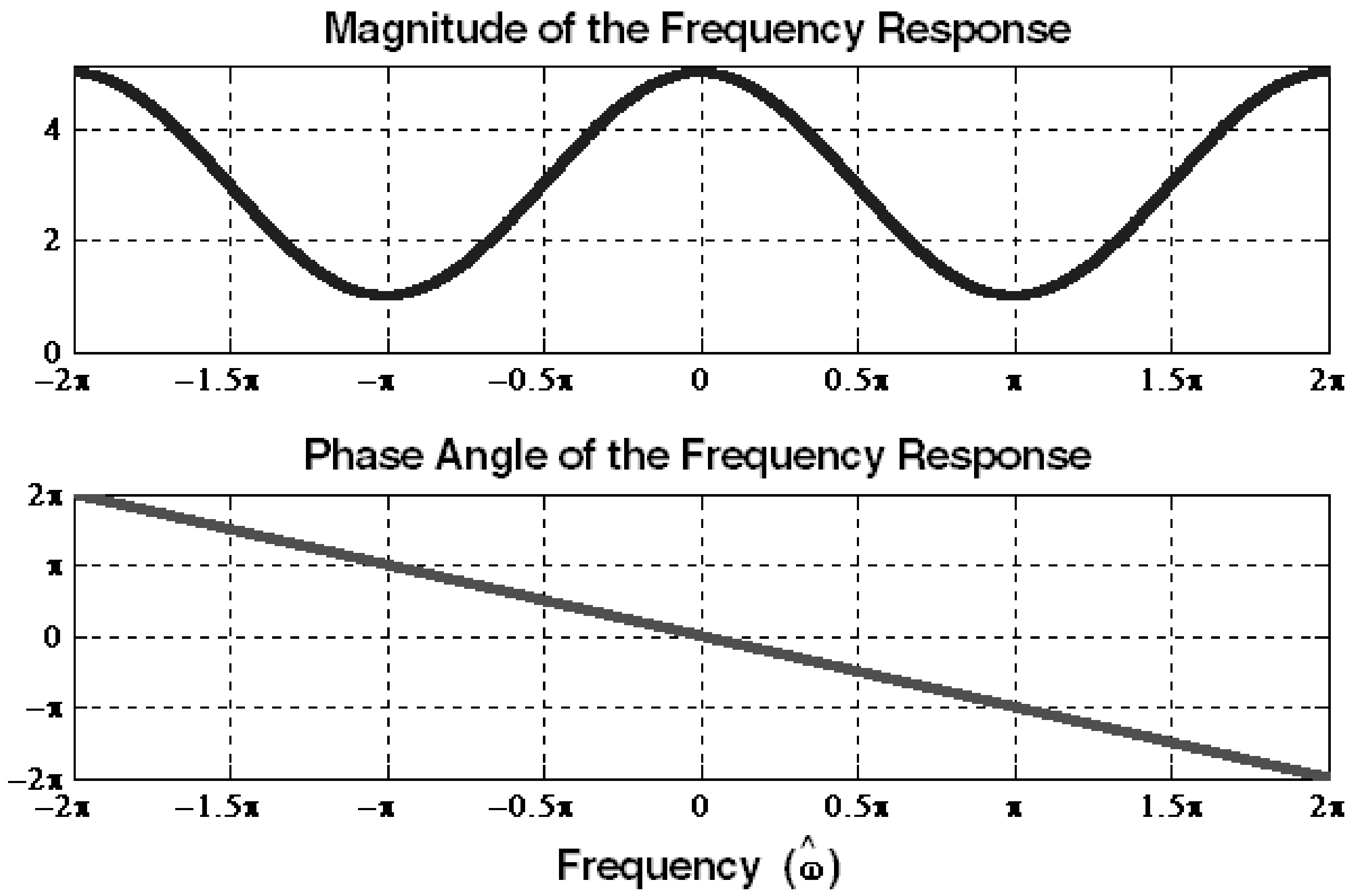
(c) Simplify first

$$\mathcal{H}(\hat{\omega}) = e^{-j\hat{\omega}}(e^{j\hat{\omega}} + 3 + e^{-j\hat{\omega}}) = (3 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

$$|\mathcal{H}(\hat{\omega})| = 3 + 2\cos\hat{\omega} \leftarrow \text{Magnitude}$$

$$\angle\mathcal{H}(\hat{\omega}) = -\hat{\omega} \leftarrow \text{phase}$$

Prob 7.2 (cont)



(d) $x_1[n] = 2\cos(0.75\pi n) = e^{j0.75\pi n} + e^{-j0.75\pi n}$

Find the output when the input is $e^{j0.75\pi n}$

$$y_{11}[n] = \mathcal{H}(0.75\pi)e^{j0.75\pi n}$$
$$= (3+2\cos(0.75\pi))e^{-j0.75\pi}e^{j0.75\pi n}$$
$$= (3-\sqrt{2})\,e^{j0.75\pi(n-1)}$$

Find the output when the input is $e^{-j0.75\pi n}$

$$y_{12}[n] = \mathcal{H}(-0.75\pi)e^{-j0.75\pi n}$$
$$= (3+2\cos(-0.75\pi))e^{+j0.75\pi}e^{-j0.75\pi n}$$
$$= (3-\sqrt{2})\,e^{-j0.75\pi(n-1)}$$

By linearity, we can then add $y_{11}[n] + y_{12}[n]$ to get $y_1[n]$.

$$y_1[n] = (3-\sqrt{2})e^{j0.75\pi(n-1)} + (3-\sqrt{2})e^{-j0.75\pi(n-1)}$$
$$= \underbrace{2(3-\sqrt{2})}_{\approx 3.1716}\cos(0.75\pi(n-1))$$

(e) $x_2[n] = 4 + 4\cos(0.75\pi(n-1))$

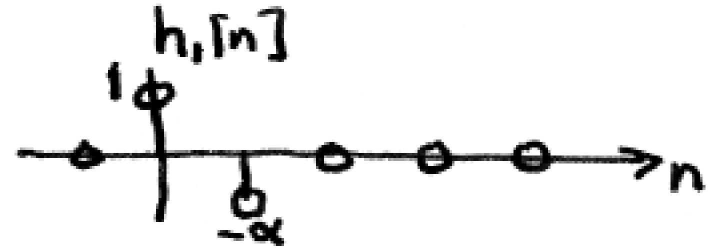
when the input is 4, the output is $4\cdot\mathcal{H}(0) = 4\cdot 5 = 20$

when the input is $4\cos(0.75\pi(n-1))$, we use part (d) to get $4(3-\sqrt{2})\cos(0.75\pi(n-2))$

$$\Rightarrow y_2[n] = 20 + 4(3-\sqrt{2})\cos(0.75\pi(n-2))$$

Prob 7.3

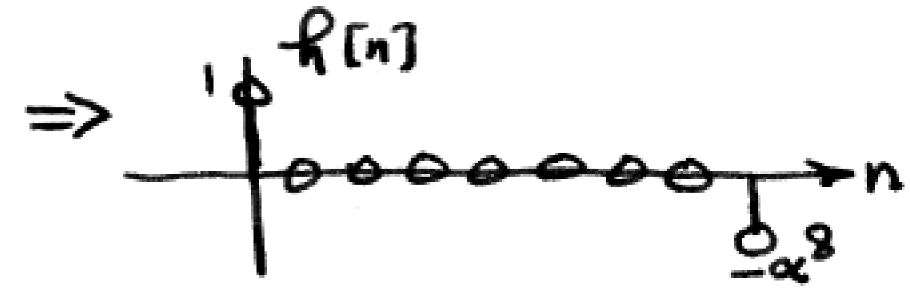
(a) $h_1[n] = \delta[n] - \alpha\delta[n-1]$



(b) Convolution:

$$
\begin{array}{ccccccccc}
1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 & \alpha^7 & \\
1 & -\alpha & & & & & & & \\
\hline
1 & \alpha & \alpha^2 & \alpha^3 & \alpha^4 & \alpha^5 & \alpha^6 & \alpha^7 & \\
 & -\alpha & -\alpha^2 & -\alpha^3 & -\alpha^4 & -\alpha^5 & -\alpha^6 & -\alpha^7 & -\alpha^8 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\alpha^8 \\
\uparrow & & & & \uparrow & & & & \uparrow \\
n=0 & & & & n=4 & & & & n=8
\end{array}
$$

$\Rightarrow$



(c) For general values of L, $h[n] = \delta[n] - \alpha^L\delta[n-L]$

(d) It is easy to convert $h[n]$ back into a difference equation.

$b_0 = 1$, $b_L = -\alpha^L$ $\Rightarrow$ $y[n] = x[n] - \alpha^L x[n-L]$

(e) If $0<\alpha<1$, then $\alpha^L \to 0$ as $L \to \infty$

So we should choose $L=\infty$ to get $y[n] = x[n]$.

Prob 7.4

(a) $\mathcal{H}_1(\hat{\omega}) = 1 - e^{-j\hat{\omega}}$ because $h_1[n] = \delta[n] - \delta[n-1] \Rightarrow b_0 = 1$ & $b_1 = -1$.

(b) For $h_2[n] = u[n] - u[n-8]$, $b_k = 1$ for $n = 0, 1, 2, \ldots 7$

$$\mathcal{H}_2(\hat{\omega}) = \sum_{k=0}^{7} b_k e^{-jk\hat{\omega}} = 1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}} + e^{-j4\hat{\omega}} + e^{-j5\hat{\omega}} + e^{-j6\hat{\omega}} + e^{-j7\hat{\omega}}$$

(c) Convolution:

$$\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
1 & -1 & & & & & & & \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \\
 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\uparrow & & & & \uparrow & & & & \uparrow \\
n=0 & & & & n=4 & & & & n=8
\end{array}$$

$\Rightarrow h[n] = \delta[n] - \delta[n-8]$

(d) $h[n] = \delta[n] - \delta[n-8] \Rightarrow b_0 = 1$ & $b_8 = -1$.

$$\mathcal{H}(\hat{\omega}) = 1 - e^{-j8\hat{\omega}}$$

(e) Now do the multiplication of $\mathcal{H}_1(\hat{\omega})\mathcal{H}_2(\hat{\omega})$

$$(1 - e^{-j\hat{\omega}})\mathcal{H}_2(\hat{\omega}) = 1 + e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} + e^{-j3\hat{\omega}} + e^{-j4\hat{\omega}} + e^{-j5\hat{\omega}} + e^{-j6\hat{\omega}} + e^{-j7\hat{\omega}} +$$
$$- e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}} - e^{-j4\hat{\omega}} - e^{-j5\hat{\omega}} - e^{-j6\hat{\omega}} - e^{-j7\hat{\omega}} - e^{-j8\hat{\omega}}$$

Therefore, $(1 - e^{-j\hat{\omega}})\mathcal{H}_2(\hat{\omega}) = 1 - e^{-j8\hat{\omega}}$ which matches part (d).

Prob 7.5

(a) $\mathcal{H}_1(\hat{\omega}) = e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} \Rightarrow b_1 = 1,\ b_2 = 1$ & $b_0 = 0$

$y_1[n] = x_1[n-1] + x_2[n-2]$

(b) For S_2, $b_0 = 1$, $b_1 = 0$, $b_2 = -1$. $\quad \mathcal{H}_2(\hat{\omega}) = 1 - e^{-j2\hat{\omega}}$

For S_3, $b_4 = 5$ & $b_5 = 5 \Rightarrow \mathcal{H}_3(\hat{\omega}) = 5e^{-j4\hat{\omega}} + 5e^{-j5\hat{\omega}}$

(c) Overall Frequency Response is the product

$$\mathcal{H}_1(\hat{\omega})\mathcal{H}_2(\hat{\omega})\mathcal{H}_3(\hat{\omega}) = (e^{-j\hat{\omega}} + e^{-j2\hat{\omega}})(1 - e^{-j2\hat{\omega}})(5e^{-j4\hat{\omega}} + 5e^{-j5\hat{\omega}})$$

Simplify:
$$= e^{-j\hat{\omega}} \cdot 5e^{-j4\hat{\omega}}(1 + e^{-j\hat{\omega}})(1 - e^{-j2\hat{\omega}})(1 + e^{-j\hat{\omega}})$$
$$= 5e^{-j5\hat{\omega}}(1 + 2e^{-j\hat{\omega}} - 2e^{-j3\hat{\omega}} - e^{-j4\hat{\omega}})$$

(d) One more step with $\mathcal{H}(\omega)$ and then we can get the $\{b_k\}$

$$\mathcal{H}(\hat{\omega}) = \underbrace{5}_{b_5}e^{-j5\hat{\omega}} + \underbrace{10}_{b_6}e^{-j6\hat{\omega}} \underbrace{-10}_{b_8}e^{-j8\hat{\omega}} \underbrace{-5}_{b_9}e^{-j9\hat{\omega}}$$

$$y[n] = 5x[n-5] + 10x[n-6] - 10x[n-8] - 5x[n-9]$$

Prob 7.6

(a) From bb = ones(1,6)/6; we have $b_k = 1/6$ for $k = 0, 1, 2, 3, 4, 5$

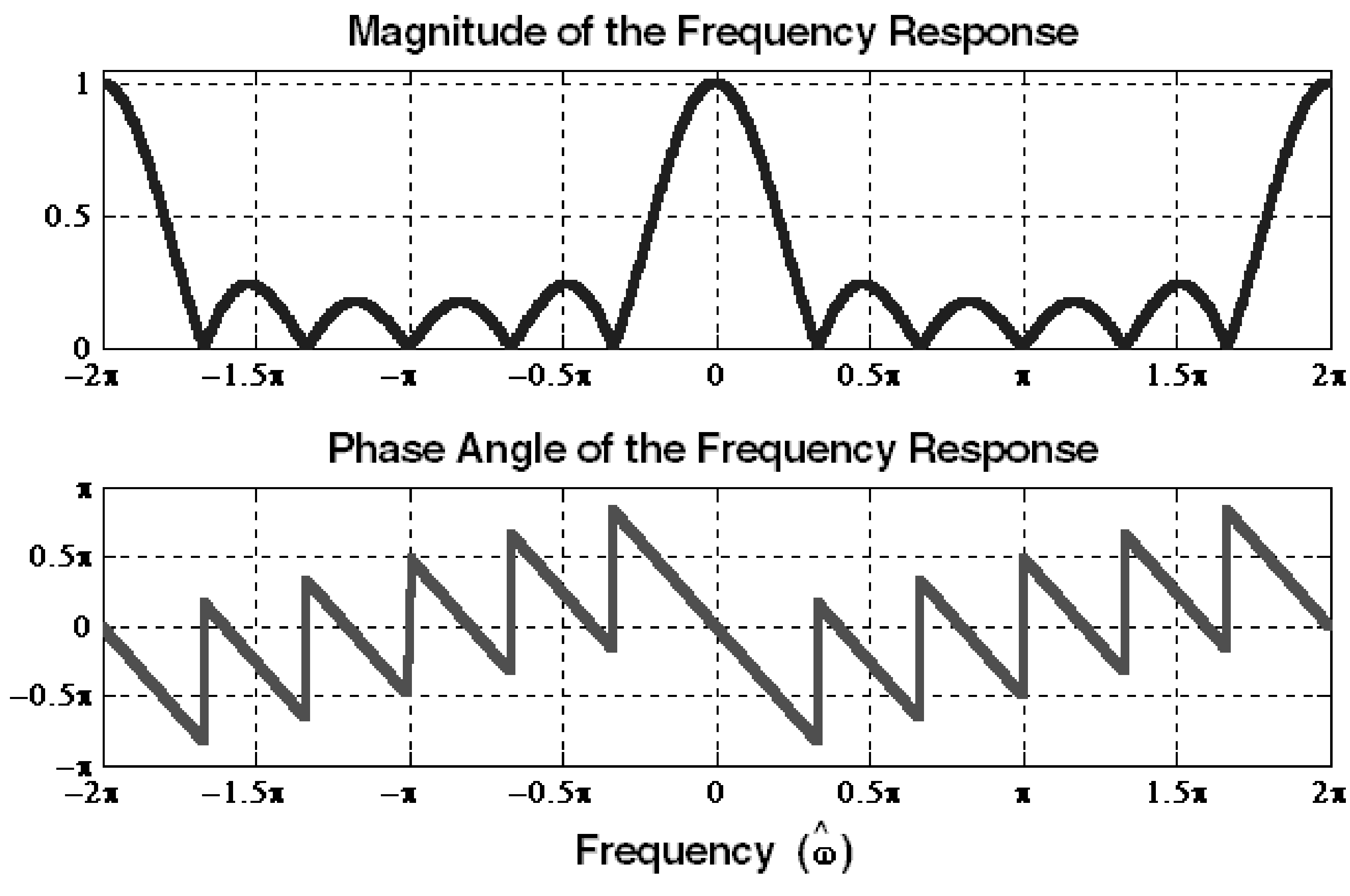
$$\mathcal{H}(\hat{\omega}) = \sum_{k=0}^{5} \frac{1}{6} e^{-jk\hat{\omega}} = \frac{1}{6} \frac{1 - e^{-j6\hat{\omega}}}{1 - e^{-j\hat{\omega}}}$$

$$= \frac{1}{6} \frac{e^{-j3\hat{\omega}}}{e^{-j\hat{\omega}/2}} \cdot \frac{e^{j3\hat{\omega}} - e^{-j3\hat{\omega}}}{e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2}}$$

This part can be used to plot $|\mathcal{H}(\hat{\omega})|$

$$= e^{-j2.5\hat{\omega}} \frac{\sin(3\hat{\omega})}{6\sin(\hat{\omega}/2)}$$

(b)



Prob 7.6

(c) The input signal values in xn are $[0, -3, 0, 3, 0, -3, 0, 3, 0, -3 \ldots]$.

Convolution:

$$
\begin{array}{cccccccccccccc}
0 & -3 & 0 & 3 & 0 & -3 & 0 & 3 & 0 & -3 & 0 & 3 & 0 & -3 \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & & & & & & & & \\
\hline
0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\
 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
 & & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\
 & & & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\
 & & & & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\
 & & & & & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\
\hline
0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \;\ldots \\
\uparrow n=0 & & & & & \uparrow n=5 & & & & & \uparrow n=10 & & &
\end{array}
$$

Notice that for $n \geq 5$, the period is 4, so the frequency is $\frac{2\pi}{4} = \frac{\pi}{2}$.
If we use the input $3\cos(\frac{\pi}{2}n + \frac{\pi}{2})$, then the output for $n \geq 5$ can be obtained by getting the magnitude & phase change from $\mathcal{H}(\pi/2) = e^{-j2.5\pi/2} \frac{\sin(3\pi/2)}{6\sin(\pi/4)} = e^{-j1.25\pi} \frac{-1}{3\sqrt{2}} = \frac{1}{3\sqrt{2}} e^{-j0.25\pi}$

$$\Rightarrow y[n] = 3\left(\tfrac{1}{3\sqrt{2}}\right) \cos\left(\tfrac{\pi}{2}n + 0.5\pi - 0.25\pi\right) \quad \text{for } n \geq 5$$

$$= \tfrac{1}{\sqrt{2}} \cos\left(\tfrac{\pi}{2}n + 0.25\pi\right) \quad \text{for } n \geq 5$$

(d) The frequency response $\mathcal{H}(\hat{\omega}) = e^{-j2.5\hat{\omega}} \frac{\sin 3\hat{\omega}}{\sin(\hat{\omega}/2)}$ is zero for $\hat{\omega} = \frac{2\pi k}{6}$, $k = 1, 2, 3, 4, 5$. Thus, all of these freqs are nulled.

$\hat{\omega} = \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$ or $\frac{5\pi}{3}$ $\leftarrow$ If the input frequency is one of these, the output will be zero, for $n \geq 5$

Prob 7.7

(a) To write the difference equation we need the b_k's.

Expand $\mathcal{H}(\hat{\omega})$ into a polynomial

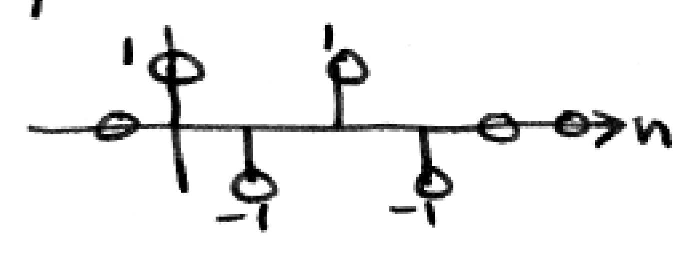
$$\mathcal{H}(\hat{\omega}) = (1-e^{-j\hat{\omega}})\underbrace{(1-je^{-j\hat{\omega}})(1+je^{-j\hat{\omega}})}_{(1 \cancel{-je^{-j\hat{\omega}}} \cancel{+je^{-j\hat{\omega}}} + e^{-j2\hat{\omega}})}$$

$$= 1 - e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}} \quad \Rightarrow \quad b_0=1,\ b_1=-1,\ b_2=1,\ b_3=-1$$

$$y[n] = x[n] - x[n-1] + x[n-2] - x[n-3]$$

(b) Use $x[n] = \delta[n]$ to get the impulse response.

$$h[n] = \delta[n] - \delta[n-1] + \delta[n-2] - \delta[n-3]$$



(c) $x[n] = Ae^{j\varphi}e^{j\hat{\omega}n}$ gives the output $y[n] = \mathcal{H}(\hat{\omega})Ae^{j\varphi}e^{j\hat{\omega}n}$

so we need to find where $\mathcal{H}(\hat{\omega}) = 0$.

Use the FACTORED FORM. If one factor is zero then $\mathcal{H}(\hat{\omega}) = 0$.

$$\left.\begin{array}{l} 1-e^{-j\hat{\omega}} = 0 \Rightarrow \hat{\omega} = 0 \\ 1-je^{-j\hat{\omega}} = 0 \Rightarrow e^{j\hat{\omega}} = j \Rightarrow \hat{\omega} = \pi/2 \\ 1+je^{-j\hat{\omega}} = 0 \Rightarrow e^{j\hat{\omega}} = -j \Rightarrow \hat{\omega} = -\pi/2 \end{array}\right\} \quad \hat{\omega} = 0, \tfrac{\pi}{2}, -\tfrac{\pi}{2}$$

(d) Work each term separately

$x_1[n] = 1$ is D.C. $\quad \mathcal{H}(\hat{\omega})\big|_{\hat{\omega}=0} = 0 \quad \Rightarrow \quad y_1[n] = 0$

$x_2[n] = 2\delta[n-3]$

Use linearity & time-invariance to say that the output is $h[n]$ shifted by 3 and doubled.

$$y_2[n] = 2h[n-3]$$
$$= 2\delta[n-3] - 2\delta[n-4] + 2\delta[n-5] - 2\delta[n-6]$$

$x_3[n] = 7\cos(0.5\pi n)$. $\quad$ Since $\mathcal{H}(0.5\pi) = 0 \Rightarrow y_3[n] = 0$

$$\therefore\ y[n] = y_1[n] + y_2[n] + y_3[n]$$
$$= 2h[n-3] \Rightarrow \text{plot}$$