EE-2025 Fall-2001

Lecture 6
Fourier Series Coefficients
10-Sept-2001

Web-CT Info

- Check the Bulletin Board for msgs
- **Get Ch3 with new Fourier Series**
  - PDF file posted to WebCT
  - Replacement for pp. 62-66 in Chapter 3
- Prob Set #2 due This Week
  - Solution will be posted Friday evening

Quiz #1 Info

- Quiz #1 on 17-Sept (Monday)
  - Calculator OK, and one page of notes
  - Coverage: HW #1 and #2
- Old Quizzes & Problems are linked via WebCT: "Word from Previous Semesters"
- Review on Sunday Evening
  - Check WebCT for details

Lab Info

- Lab #2 Report
  - Turn in during your lab time
  - Write-up section 4 on "Direction Finding"
  - Finish INSTRUCTOR VERIFICATION in Lab
  - ERRATA ? ALWAYS Check Bulletin Board
- Lab #3 has been posted
  - Read the Pre-Lab and do it before lab

LECTURE

READING ASSIGNMENTS

- This Lecture:
  - **Revised Chap 3 with Fourier Series**
    - PDF file available on WebCT
    - pp. 3016-3031
    - Replaces pp. 62-66 in Chapter 3
- Other Reading:
  - Next Lecture: Chap. 4 on Sampling

LECTURE OBJECTIVES

- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi k/T_0)t}dt$$

- **ANALYSIS** via Fourier Series
  - For PERIODIC signals: $\mathbf{x(t+T) = x(t)}$
  - **SPECTRUM from the Fourier Series**

HISTORY

- Jean Baptiste Joseph Fourier
  - 1807 thesis (memoir)
    - On the Propagation of Heat in Solid Bodies
  - Heat !
  - Napoleonic era
- http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html

Joseph Fourier

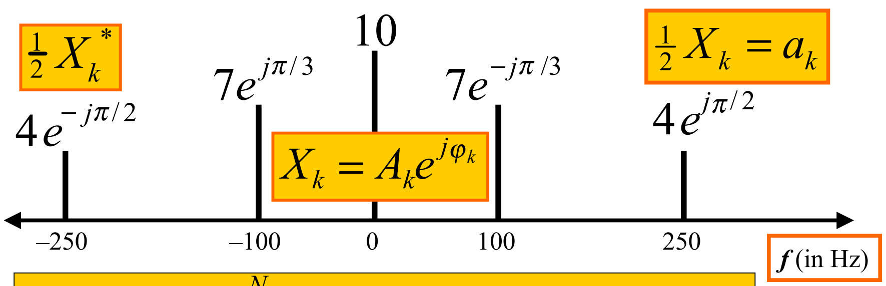
lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Find out more at:
http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fourier.html

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = a_0 + \sum_{k=1}^{N} \{ a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \}$$

Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

$$a_k = \tfrac{1}{2} X_k = \tfrac{1}{2} A_k e^{j\varphi_k}$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

COMPLEX AMPLITUDE

$$X_k = A_k e^{j\varphi_k}$$

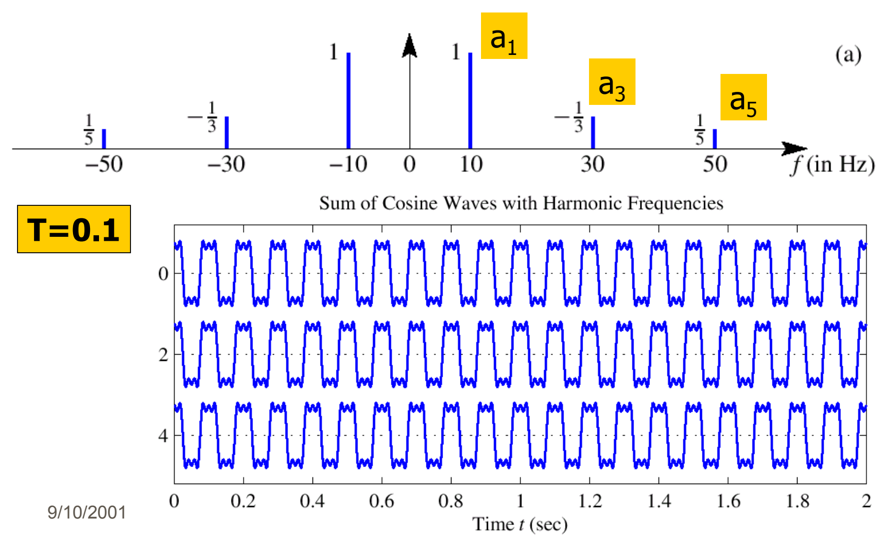
Harmonic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

Harmonic Signal (3 Freqs)



SYNTHESIS vs. ANALYSIS

- SYNTHESIS
  - **Easy**
  - Given (ω_k,A_k,ϕ_k) create x(t)
- Synthesis can be HARD
  - Synthesize Speech so that it sounds good
- ANALYSIS
  - **Hard**
  - Given x(t), extract (ω_k,A_k,ϕ_k)
  - How many?
  - Need algorithm for computer

STRATEGY

- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - The answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 k t}dt$$

ORTHOGONALITY of exp(j)

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j2\pi m t/T_0}dt = \frac{T_0}{-j2\pi m}e^{-j2\pi m t/T_0}\Big|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m}\left(e^{-j2\pi m} - 1\right)$$

$$\int_0^{T_0} e^{-jm\omega_0 t}dt = 0$$

$$m \neq 0$$

$$\omega_0 = \frac{2\pi}{T_0}$$

ORTHOGONALITY of exp(j)

- INTEGRATE over ONE PERIOD

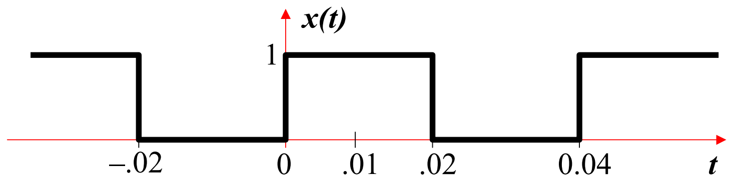
$$\frac{1}{T_0}\int_0^{T_0} e^{j2\pi\ell t/T_0}e^{-j2\pi k t/T_0}dt = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$$

$$\frac{1}{T_0}\int_0^{T_0} e^{j2\pi(\ell-k)t/T_0}dt$$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2}T_0 \\ 0 & \frac{1}{2}T_0 \le t < T_0 \end{cases}$$

for $T_0 = 0.04\,\text{sec}$:



FS for a SQUARE WAVE

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt \qquad (k \ne 0)$$

$$a_k = \frac{1}{.04}\int_0^{.02} 1e^{-j2\pi kt/.04}dt = \frac{1}{.04(-j\pi k/.02)}\, e^{-j\pi kt/.02}\Big|_0^{.02}$$

$$= \frac{1}{(-2j\pi k)}\left(e^{-j\pi k} - 1\right) = \frac{1-(-1)^k}{j2\pi k}$$

DC Coefficient, a_0

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kt/T_0}dt \qquad (k = 0)$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0}(\text{AREA})$$

$$a_0 = \frac{1}{.04}\int_0^{.02} dt = \frac{1}{.04}(.02 - 0) = \frac{1}{2}$$

Fourier Coefficients a_k

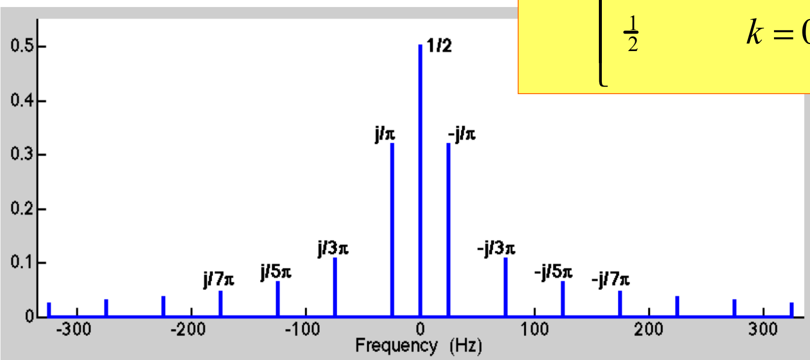
- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

$$a_k = \frac{1-e^{-j\pi k}}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

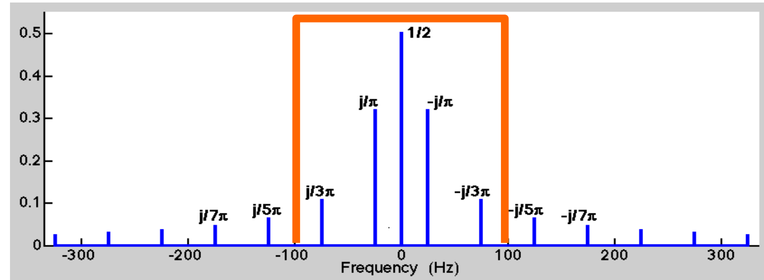
$\omega_0 = 2\pi / (0.04) = 2\pi(25)$

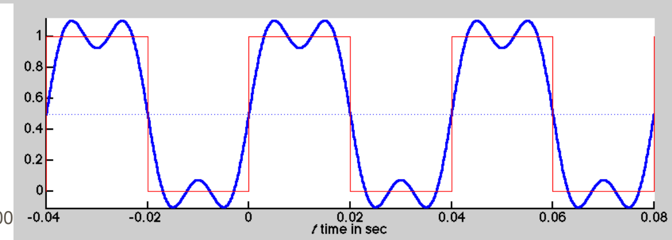
$$a_k = \begin{cases} \dfrac{-j}{\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \tfrac{1}{2} & k = 0 \end{cases}$$



Synthesis: 1st & 3rd Harmonics

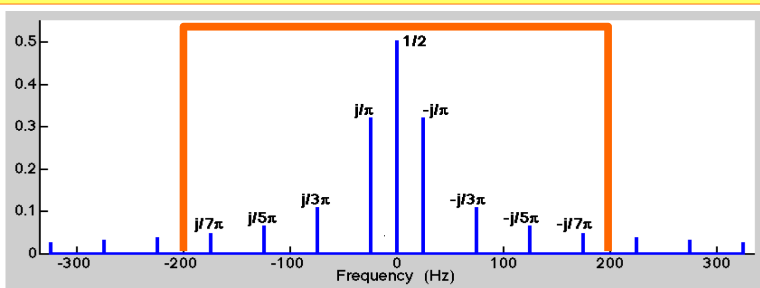
$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi(25)t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\cos(2\pi(75)t - \tfrac{\pi}{2})$$

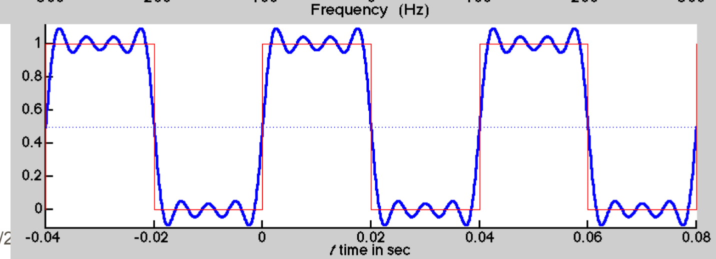




Synthesis: up to 7th Harmonic

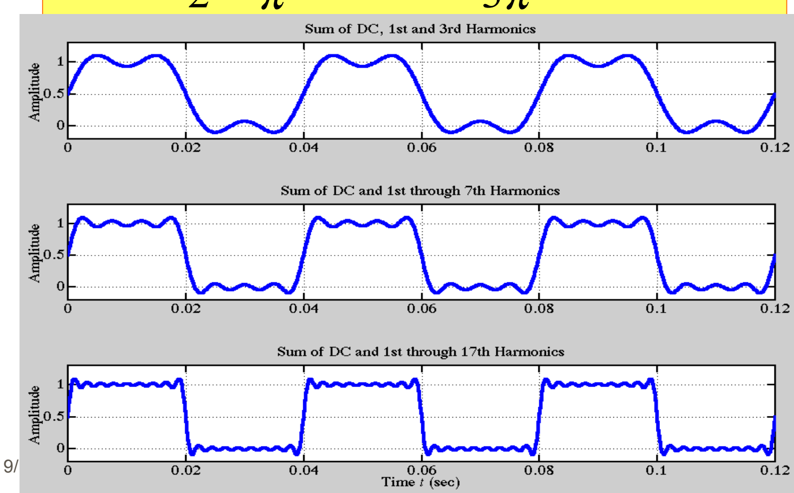
$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(50\pi t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\sin(150\pi t) + \frac{2}{5\pi}\sin(250\pi t) + \frac{2}{7\pi}\sin(350\pi t)$$





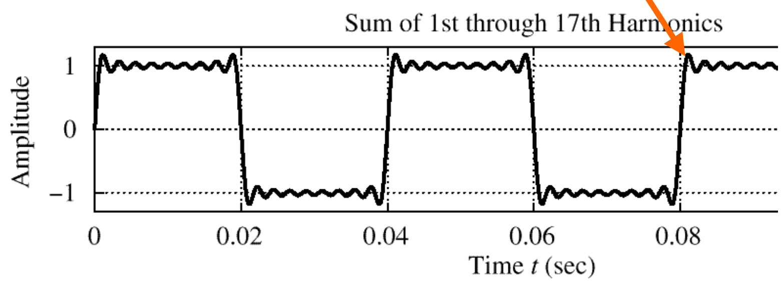
Fourier Synthesis

$$x_N(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \ldots$$



Gibbs' Phenomenon

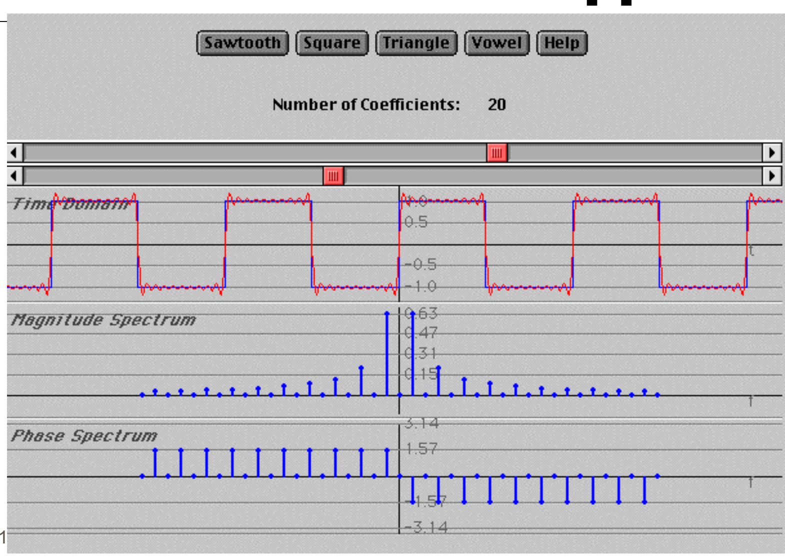
- Convergence at DISCONTINUITY of x(t)
  - There is always an overshoot
  - 9% for the Square Wave case



Fourier Series Demo

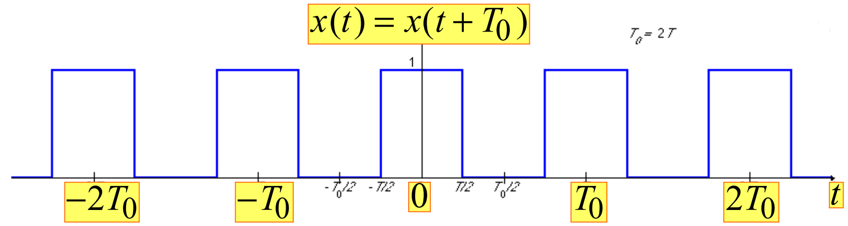
- Fourier Series Java Applet
  - Greg Slabaugh
    - Interactive
  - http://users.ece.gatech.edu/~slabaugh/java/fourier/fourier.html

Fourier Series Java Applet



General Periodic Signals

$$x(t) = x(t + T_0)$$



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

Fourier Synthesis

Fundamental Freq.

$$\omega_0 = 2\pi / T_0 = 2\pi f_0$$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

Fourier Analysis