

Lecture 18
Introduction to the Fourier Transform
5-Nov-01

Info: Web-CT, Lab, HW

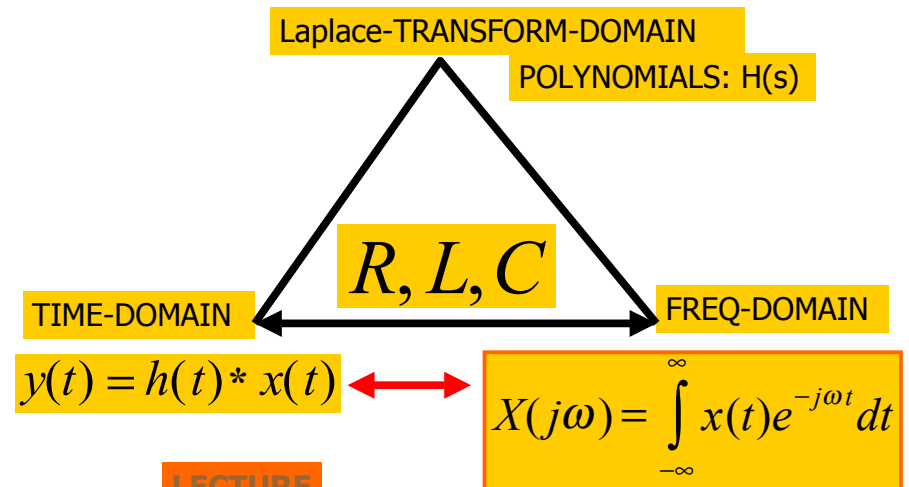
- Calendar:
 - Quiz #3 is 19-Nov (Monday)
 - Covers HW #8, #9 and #10
 - One page hand-written notes (both sides)
 - Calculator OK
- Prob Set #9 is due this week
- Labs will meet on 19 & 20-Nov
 - Recitations will meet 19, 20 & 21 Nov

Info: Lab #10

- Lab #10 is NUMERICAL FOURIER SERIES
- New MATLAB Functions:
 - `fplot.m` and `quad8.m`
 - In version 6: `quad1.m`
 - `quad8(inline('t.*t'), 0, 1)`

Evaluate $\int_0^1 t^2 dt$ to get $\frac{1}{3}$

THREE DOMAINS: ANALOG



READING ASSIGNMENTS

- This Lecture:
 - Chapter 12, pp. 1079-1093
- Other Reading:
 - Recitation: Chapter 11
 - And Chapter 12, pp. 1079-1093, 1097-1102, 1117-1118
 - Next Lecture: Chapter 12, pp. 1093-1097, 1102-1118

LECTURE OBJECTIVES

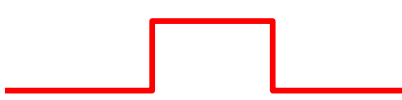
- Review
 - Frequency Response
 - Fourier Series
- Definition of **Fourier transform**

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

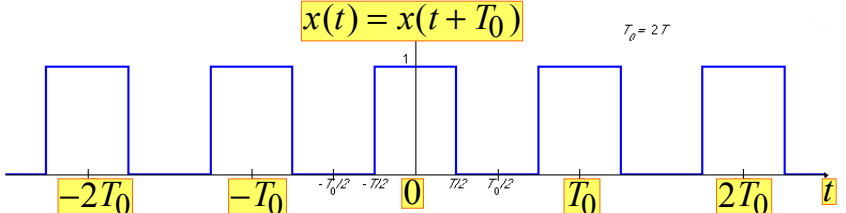
Relation to Fourier Series

- Examples of Fourier transform pairs

Everything = Sum of Sinusoids

- One Square Pulse = Sum of Sinusoids
 - ??????????????
 - Finite Length
 - Not Periodic
- 
- Limit of Square Wave as Period → infinity
 - Intuitive Argument

Fourier Series: Periodic $x(t)$



$x(t) = x(t + T_0)$

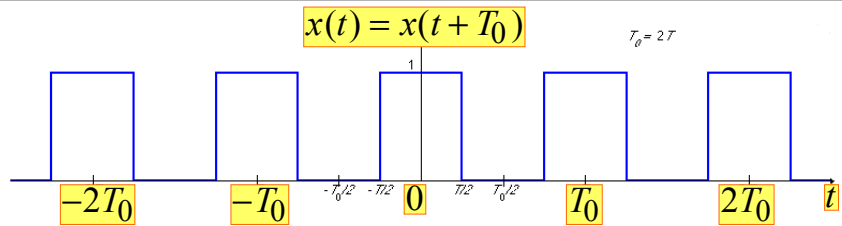
$T_0 = 2T$

$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$ ← **Fourier Synthesis**

Fundamental Freq.
 $\omega_0 = 2\pi / T_0 = 2\pi f_0$

$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j\omega_0 k t} dt$ ← **Fourier Analysis**

Square Wave Signal



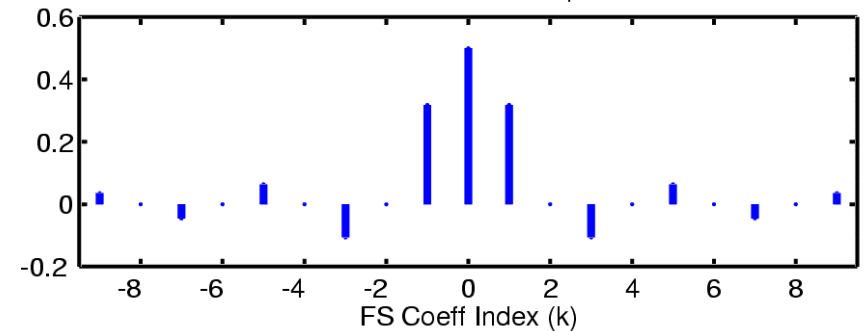
$$a_k = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} (1) e^{-j\omega_0 kt} dt$$

$$a_k = \frac{e^{-j\omega_0 k T_0/4} - e^{j\omega_0 k T_0/4}}{-j\omega_0 k T_0} = \frac{e^{-j\pi k/2} - e^{j\pi k/2}}{-j2\pi k} = \frac{\sin(\pi k / 2)}{\pi k}$$

Spectrum from Fourier Series

$$a_k = \frac{\sin(\pi k / 2)}{\pi k} = \begin{cases} \neq 0 & k = 0, \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \end{cases}$$

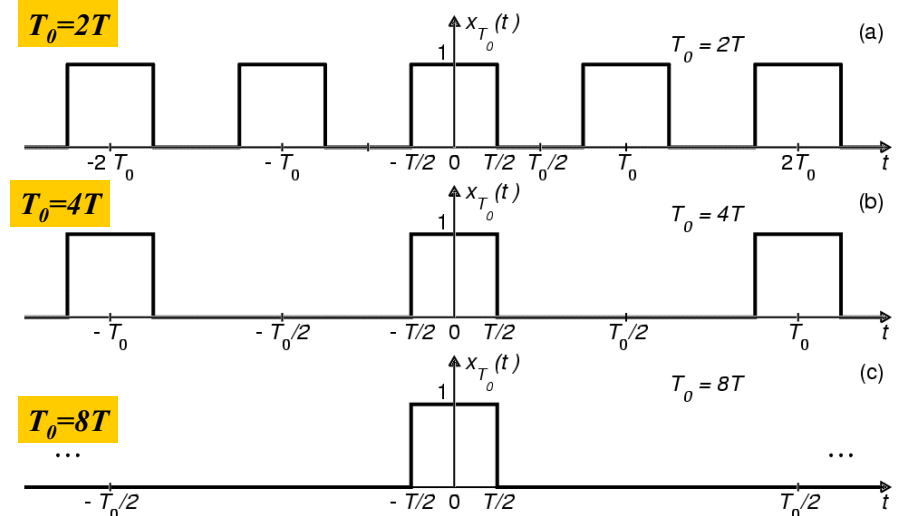
Fourier Series Coeffs for Square Wave



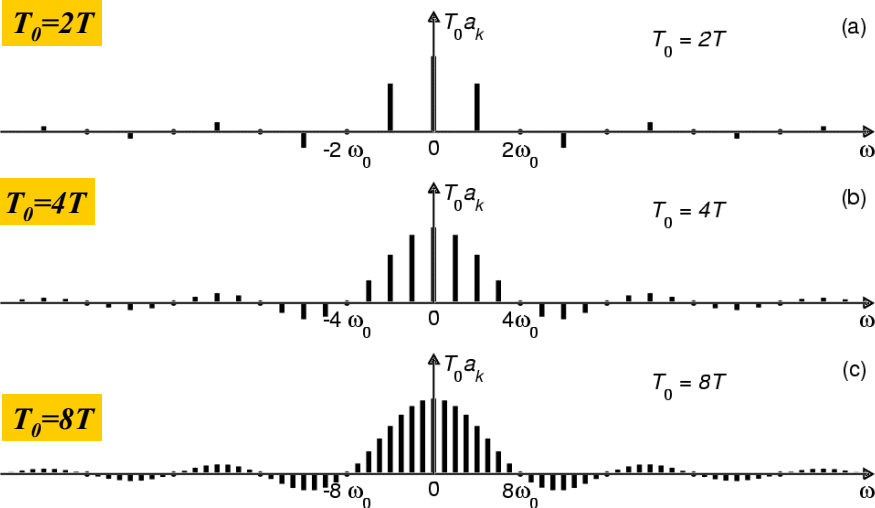
What if x(t) is not periodic?

- Sum of Sinusoids?
 - Non-harmonically related sinusoids
 - Would not be periodic, but would be non-zero for all t.
- Fourier transform
 - gives a "sum" (actually an **integral**) that involves **ALL** frequencies
 - can represent signals that are identically zero for negative t. !!!!!!!!

Limiting Behavior of FS



Limiting Behavior of Spectrum



FS in the LIMIT (long period)

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} (T_0 a_k) e^{j\omega_0 k t} \left(\frac{2\pi}{T_0} \right) \mapsto x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

$$\lim_{T_0 \rightarrow \infty} \frac{2\pi}{T_0} = d\omega \qquad \lim_{T_0 \rightarrow \infty} \frac{2\pi}{T_0} k = \omega$$

$$T_0 a_k = \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_0 k t} dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

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Fourier Transform

■ For non-periodic signals

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \quad \text{Fourier Synthesis}$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \quad \text{Fourier Analysis}$$

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Example 1:

$$x(t) = e^{-at} u(t)$$

$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$X(j\omega) = -\frac{e^{-at} e^{-j\omega t}}{a+j\omega} \Big|_0^{\infty} = \frac{1}{a+j\omega} \quad a > 0$$

$$X(j\omega) = \frac{1}{a+j\omega}$$

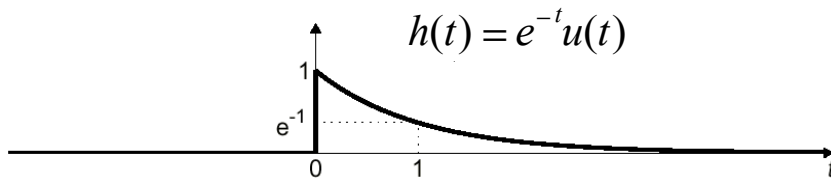
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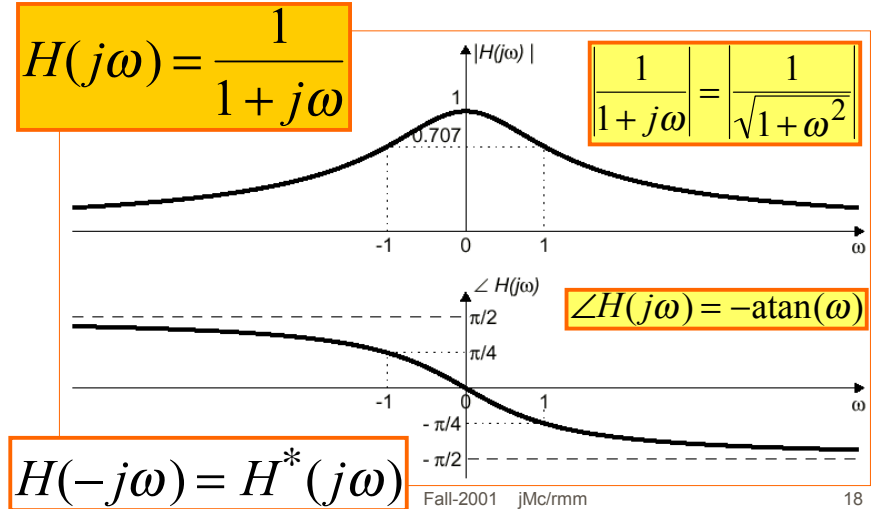
Frequency Response

- Fourier Transform of $h(t)$ is the Frequency Response



$$h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1 + j\omega}$$

Magnitude and Phase Plots



Example 2:

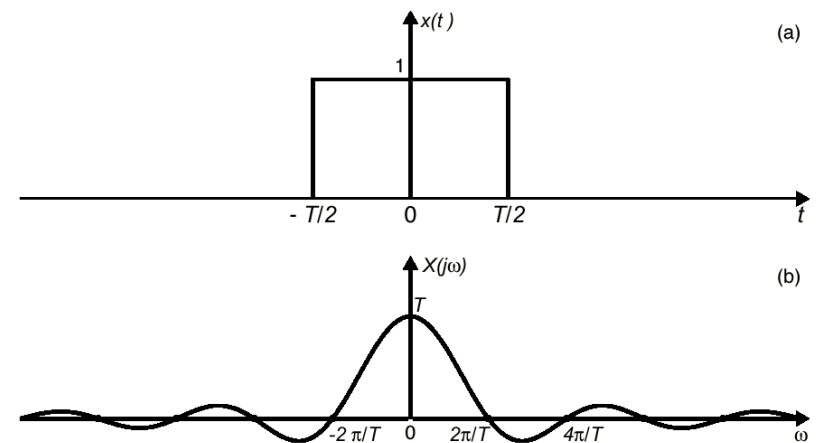
$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}$$

$$X(j\omega) = \int_{-T/2}^{T/2} (1)e^{-j\omega t} dt = \int_{-T/2}^{T/2} e^{-j\omega t} dt$$

$$X(j\omega) = \frac{e^{-j\omega t}}{-j\omega} \Big|_{-T/2}^{T/2} = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$X(j\omega) = \frac{\sin(\omega T / 2)}{(\omega / 2)}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T / 2)}{(\omega / 2)}$$



Example 3:

$$X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega t} d\omega$$

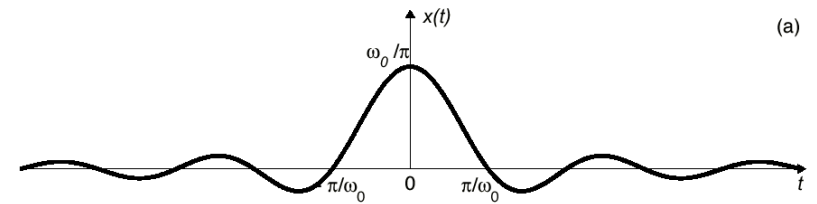
$$x(t) = \frac{1}{2\pi} \left. \frac{e^{j\omega t}}{jt} \right|_{-\omega_0}^{\omega_0} = \frac{1}{2\pi} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{jt}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)}$$

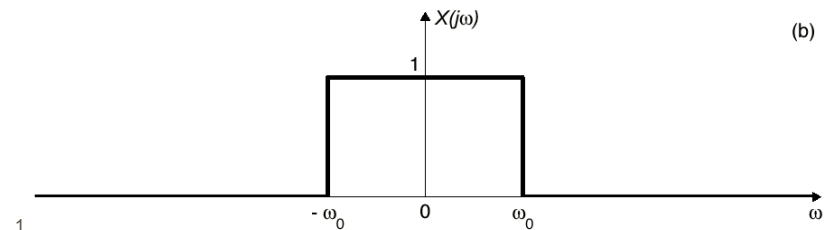
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$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$



(a)



(b)

Example 4:

$$x(t) = \delta(t)$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Shifting Property of the Impulse

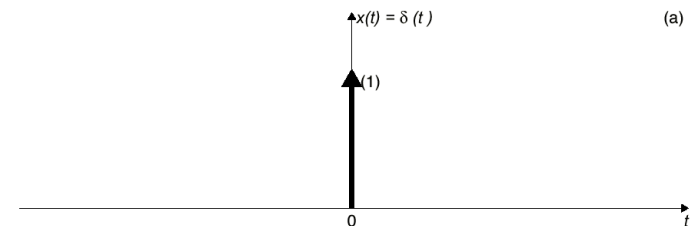
$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0}$$

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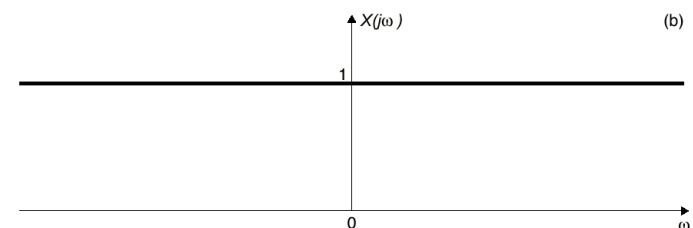
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$$x(t) = \delta(t) \Leftrightarrow X(j\omega) = 1$$



(a)



(b)

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Example 5: $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)$$

$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$

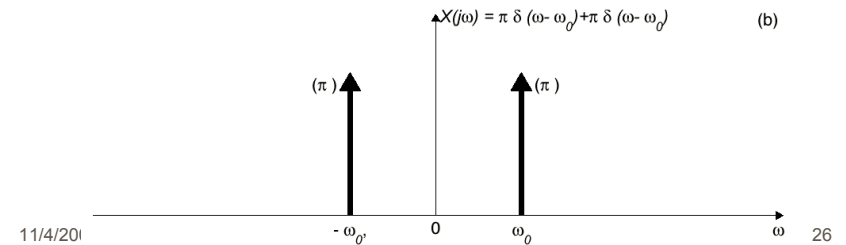
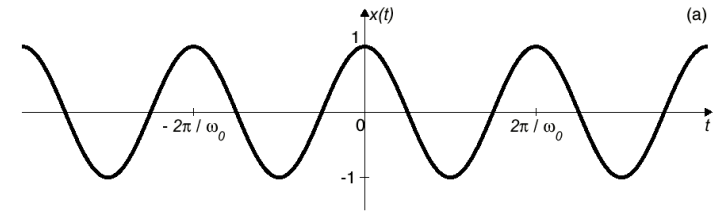
$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

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$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



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Table of Fourier Transforms

$$x(t) = e^{-at} u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

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