EE-2025 Fall-2001

Lecture 21
Sampling and Reconstruction
16-Nov-01

Info: Web-CT, Lab, HW

- **Quiz #3 is Monday, 19-Nov**
  - Covers CT signals/sys, Impulses, Convolution and Fourier Transform, AND z-Transform
  - **HW #8, #9, #10**
  - One page hand-written notes
  - Calculator is OK
- Quiz Review: Sunday (18-Nov) @ 6:30 pm
- **Labs will be held on 19 & 20 Nov**
  - Recitations on Mon, Tues & Wed

Thanksgiving & beyond

- Finals Week:
  - 11am Lecture: Thurs, 13-Dec @ 11:30am
  - Noon Lecture: Mon, 10-Dec @ 2:50pm
  - Reviews will be held
- Reading Assignment: Ch 13 of Notes.
- Prob Set #11 - due after Holiday
- Lab #11 due week of 26-Nov
  - Lab #12 due the last week (3 — 6 Dec)
  - ALL Labs must be turned in by 7-Dec-00

Pop Quiz

$$e^{-2t}u(t) * \delta(t+1) = ?$$

$$\left(e^{-2t}u(t)\right)\delta(t+1) = ?$$

$$\int_{-\infty}^{\infty}\left(e^{-2t}u(t)\right)\delta(t-1)dt = ?$$

LECTURE OBJECTIVES

- Sampling Theorem Revisited
  - GENERAL: in the FREQUENCY DOMAIN
  - Fourier transform of sampled signal
  - Reconstruction from samples
- Review of FT properties
  - Convolution <--> multiplication
  - Frequency shifting
  - Review of AM

Table of FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

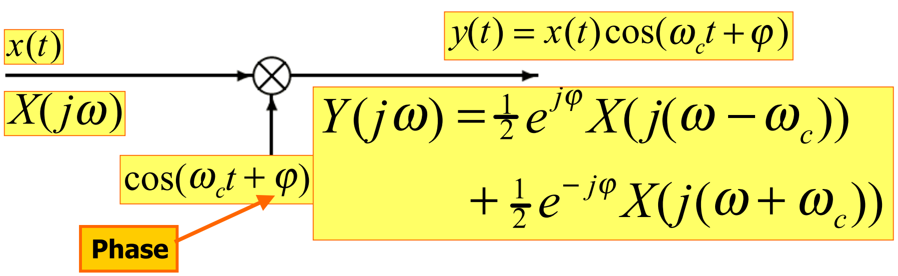
Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Scaling

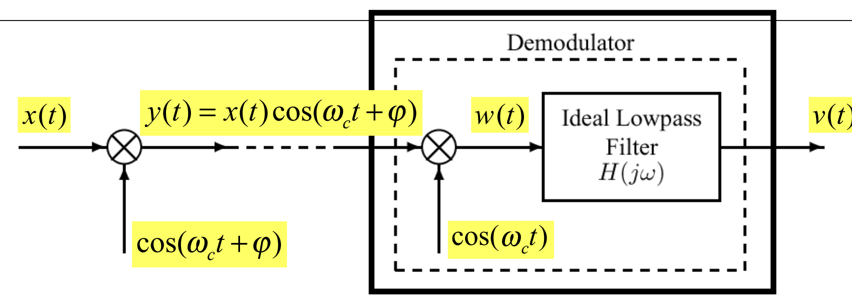
$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\frac{\omega}{a}))$$

Amplitude Modulator



- x(t) modulates the amplitude of the cosine wave. The result in the frequency-domain is two SHIFTED copies of X(jω).

DSBAM Demod Phase Synch



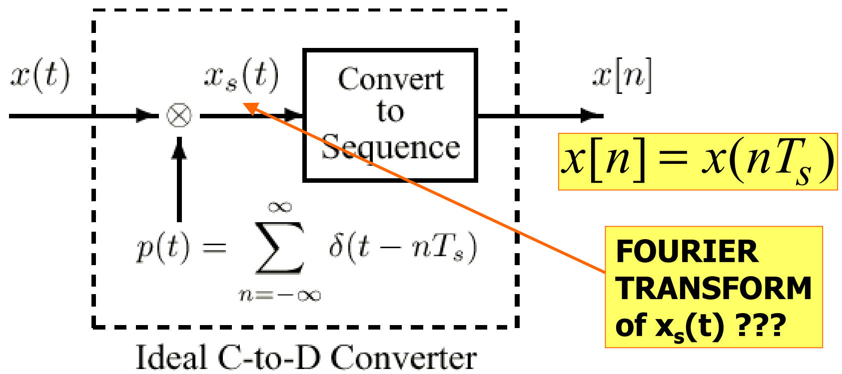
$= \frac{1}{2}\cos(\varphi)X(j\omega)$ what if $\varphi = \frac{1}{2}\pi$?

$$W(j\omega) = \frac{1}{4} e^{j\varphi} X(j\omega) + \frac{1}{4} e^{-j\varphi} X(j\omega)$$

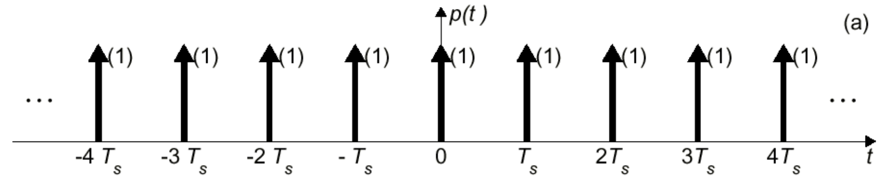
$$+ \frac{1}{4} e^{j\varphi} X(j(\omega - 2\omega_c)) + \frac{1}{4} e^{-j\varphi} X(j(\omega + 2\omega_c))$$

Ideal C-to-D Converter

- **Mathematical Model for A-to-D**



Periodic Impulse Train

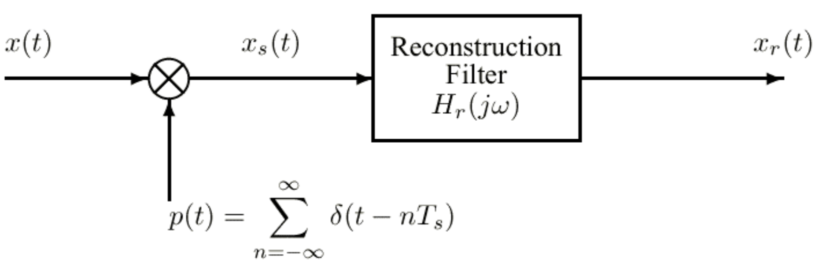


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T_s}$$

$$\omega_s = \frac{2\pi}{T_s}$$

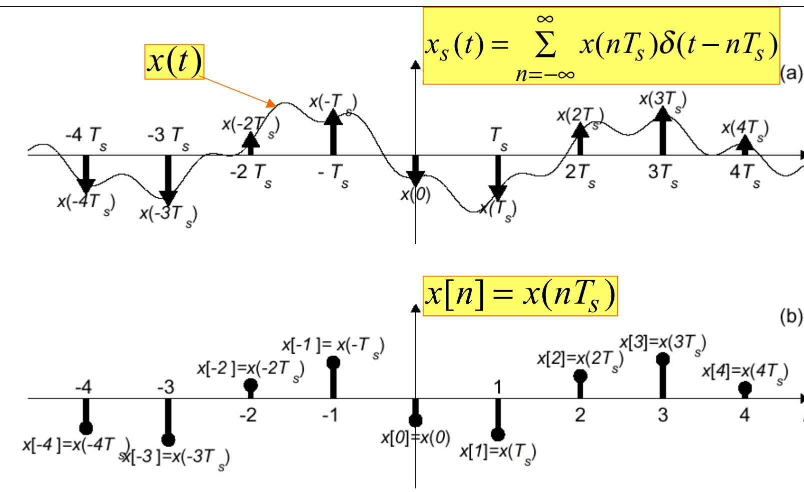
Impulse Train Sampling



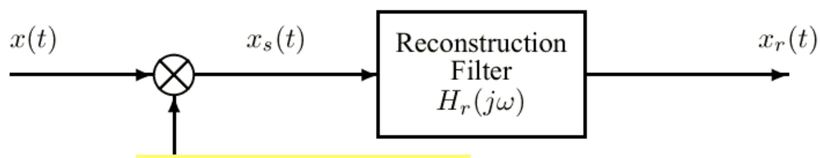
$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT_s)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Illustration of Sampling



Sampling: Freq. Domain



$$p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

EXPECT FREQUENCY SHIFTING !!!

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

Frequency-Domain Analysis

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

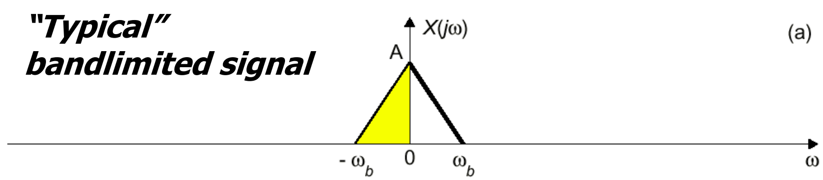
$$x_s(t) = x(t) \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} x(t) e^{jk\omega_s t}$$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

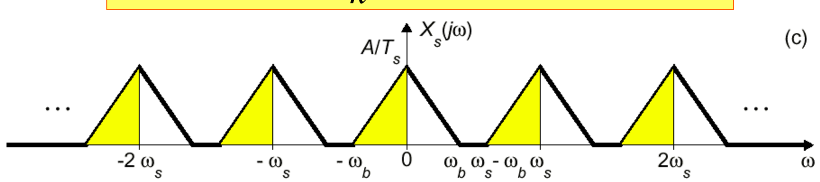
$$\omega_s = \frac{2\pi}{T_s}$$

Frequency-Domain Representation of Sampling

"Typical" bandlimited signal

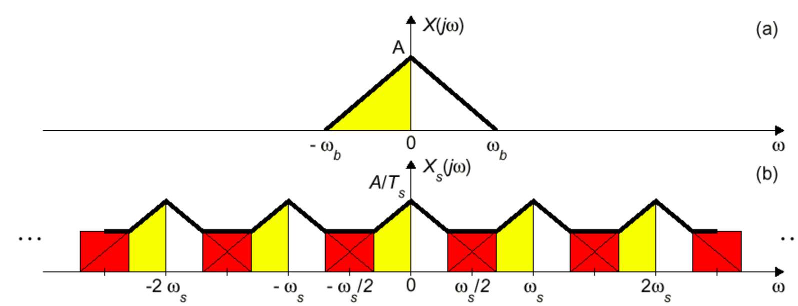


$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

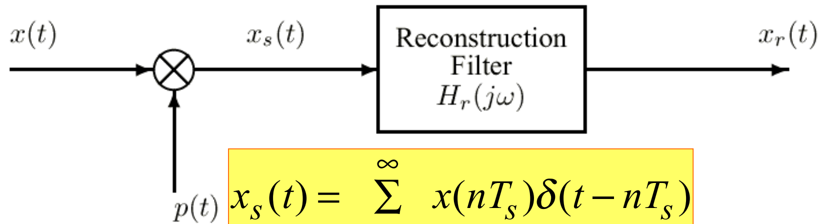


Aliasing Distortion

- If $\omega_s < 2\omega_b$, the copies of $X(j\omega)$ overlap, and we have aliasing distortion.



Reconstruction of *x*(*t*)

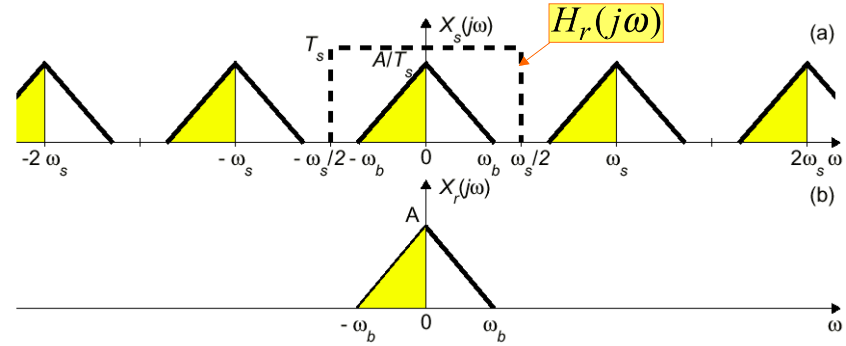


$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega)X_s(j\omega)$$

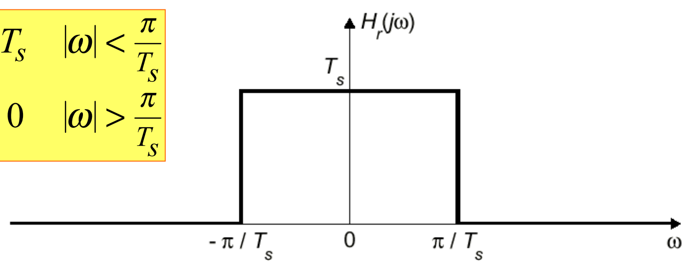
Reconstruction in the Frequency-Domain



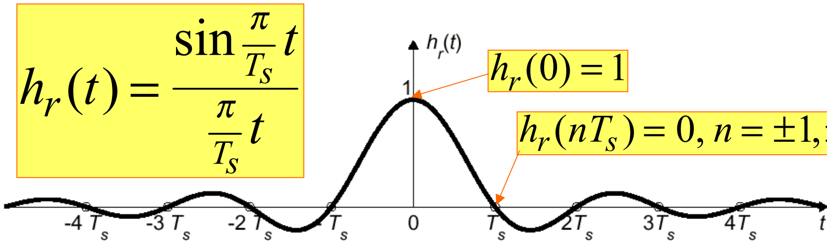
- If $\omega_s > 2\omega_b$, the copies of $X(j\omega)$ do not overlap, so $X_r(j\omega) = H_r(j\omega)\, X_s(j\omega)$.

Ideal Reconstruction Filter

$$H_r(j\omega) = \begin{cases} T_s & |\omega| < \frac{\pi}{T_s} \\ 0 & |\omega| > \frac{\pi}{T_s} \end{cases}$$



$$h_r(t) = \frac{\sin\frac{\pi}{T_s}t}{\frac{\pi}{T_s}t}$$



Signal Reconstruction

$$x_r(t) = h_r(t) * x_s(t) = h_r(t) * \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) h_r(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\frac{\sin\frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolation formula

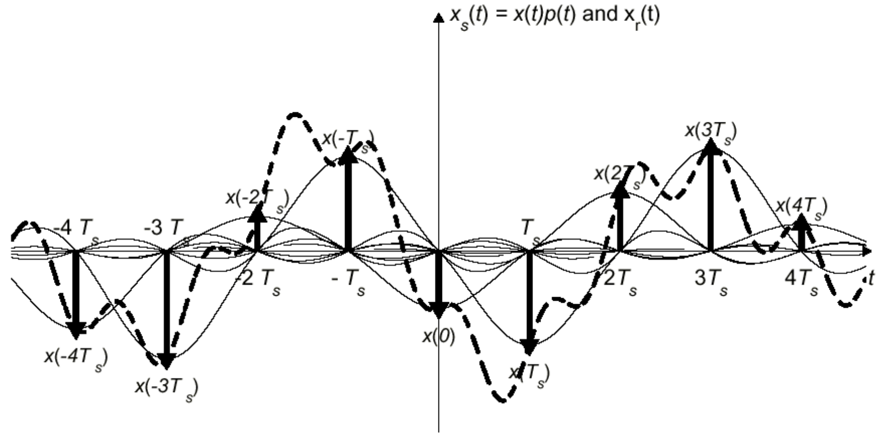
Shannon Sampling Theorem

- **"SINC" Interpolation** is the ideal
  - PERFECT RECONSTRUCTION
  - of BANDLIMITED SIGNALS

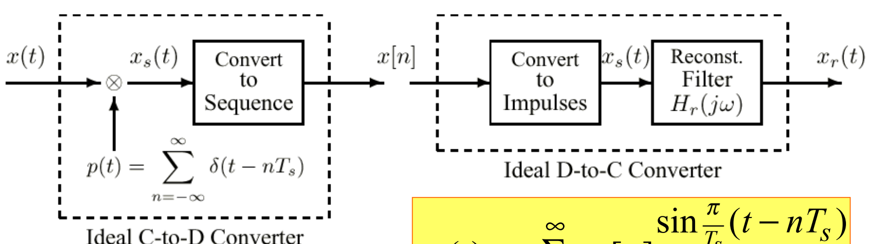
A signal $x(t)$ with bandlimited Fourier transform such that $X(j\omega) = 0$ for $|\omega| \geq \omega_b$ can be reconstructed exactly from samples taken with sampling rate $\omega_s = 2\pi/T_s \geq 2\omega_b$ using the following bandlimited interpolation formula:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin\left[\frac{\pi}{T_s}(t - nT_s)\right]}{\frac{\pi}{T_s}(t - nT_s)}.$$

Reconstruction in Time-Domain



Ideal C-to-D and D-to-C



$$x[n] = x(nT_s)$$

Ideal Sampler

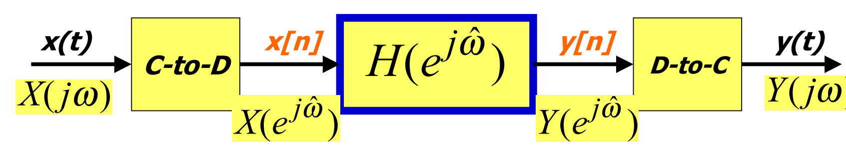
$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin\frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolator

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega) X_s(j\omega)$$

DT Filtering of CT Signals

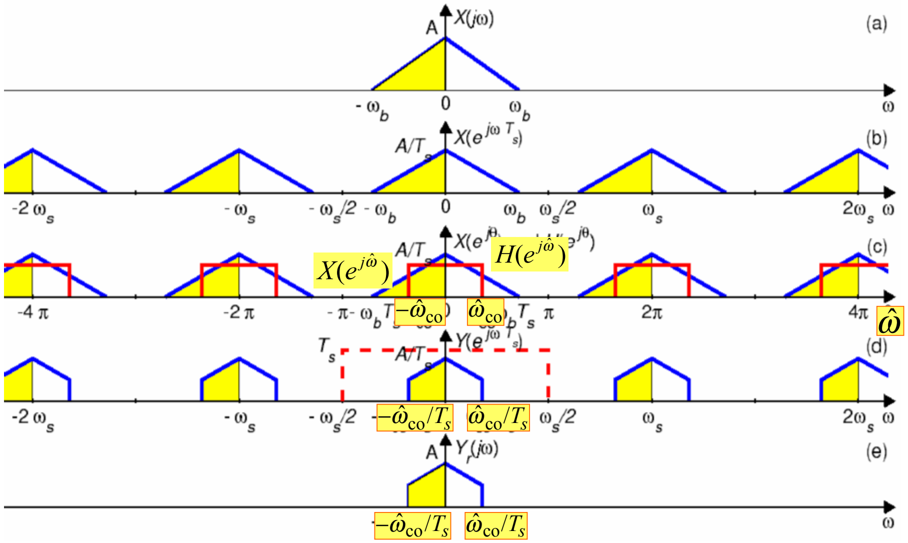


If no aliasing occurs in sampling x(t), then it follows that

$$Y(j\omega) = H_{\text{eff}}(j\omega) X(j\omega)$$

$$H_{\text{eff}}(j\omega) = \begin{cases} H(e^{j\omega T_s}) & |\omega| < \frac{1}{2}\omega_s \\ 0 & |\omega| > \frac{1}{2}\omega_s \end{cases}$$

Illustration of DT Filtering of a CT Signal



EFFECTIVE Freq. Response

- Assume NO Aliasing, then
  - ANALOG FREQ <--> DIGITAL FREQ

$$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$$

- So, we can plot:
- Scaled Freq. Axis

DIGITAL FILTER

$$H(e^{j\omega T_s}) \text{ vs. } \omega$$

ANALOG FREQUENCY