

EE-2025

Spring-2003

Lecture 6

**Fourier Series Analysis
27-Jan-2003**

General Info

- **Help Sessions: M1, M6, T6 and W4:30**
 - Every week
- **Bulletin Board: OFFICIAL ANNOUNCEMENTS**
 - Old Quizzes & Problems are linked via WebCT
- **New Ch3 has new Fourier Series**
 - PDF file posted to WebCT
 - Replaces pp. 62-66 in DSP First Ch 3
- Prob Set #3 due This Week
 - **Solution will be posted Thurs at 5pm**

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Lab Info

- Lab #2 Report
 - Turn in during your lab time
 - Late? –10 points per day
 - Finish INSTRUCTOR VERIFICATION in Lab
 - Come prepared with some preliminary code
 - **ERRATA ? ALWAYS Check the Bulletin Board**
- Lab #3 has been posted
 - **Bring Headphones to lab for the next few weeks**
 - Read the Pre-Lab and do it before lab

Quiz #1 Info

- **Quiz #1 on 31-Jan-03 (Friday)**
 - Coverage: HW #1, #2 and #3
 - Allowed one page of notes (Handwritten)
 - Review Session:
 - 7:30 pm Thursday, 30-Jan, in ECE Auditorium
 - Check WebCT for other related announcements
- Old Quizzes & Problems are linked via WebCT:
 - **“Word from Previous Semesters”**

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The Rules

- Quizzes
 - NO make-ups given
 - Next Quiz would count for the one missed, IF excused
- Excused Absence
 - Must be written (by an “official”)
 - Notify ahead of time via e-mail
- Consult “INFO” on Web-CT for more details
 - Late Labs are –10 points per day
 - No late Homework

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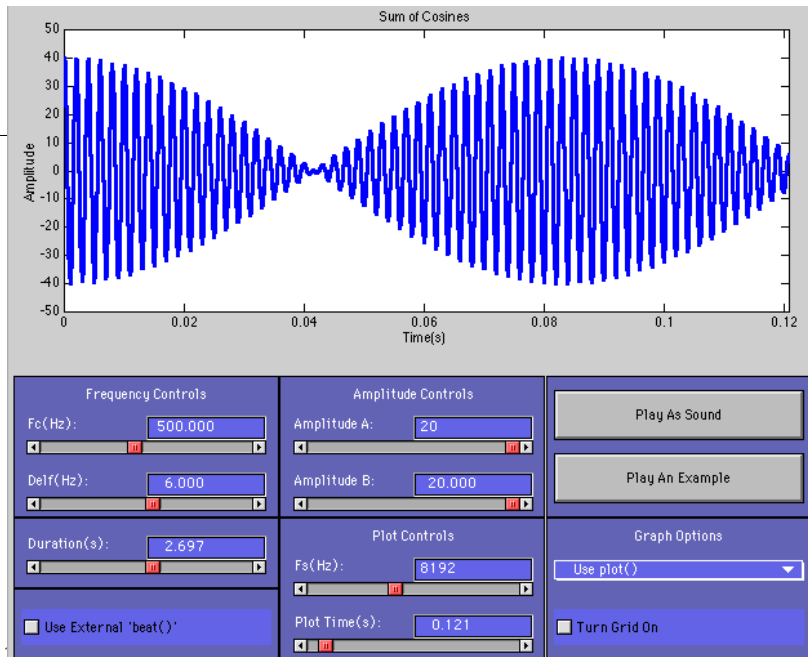
FYI: Demo on CD-ROM

- Beat Control GUI
 - Called `beatcon.m`
 - Found in SP-First Toolbox
 - Install SP-First toolbox
 - Must be on MATLAB's path
 - Use the `path` or `addpath` commands

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INSTANTANEOUS FREQ of the Chirp

- Chirp Signals have Quadratic phase
- Freq will change **LINEARLY** vs. time

$$x(t) = A \cos(\alpha t^2 + \beta t + \varphi)$$

$$\Rightarrow \psi(t) = \alpha t^2 + \beta t + \varphi$$

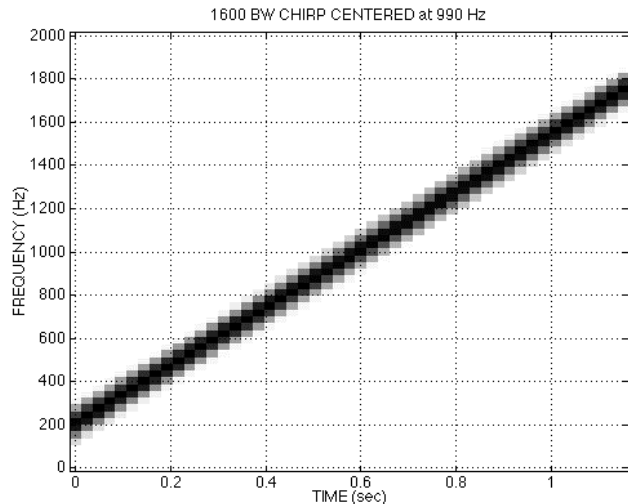
$$\Rightarrow \omega_i(t) = \frac{d}{dt} \psi(t) = 2\alpha t + \beta$$

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CHIRP SPECTROGRAM



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Question: Create a Chirp

- Chirp should start at 200 Hz and end at 3200 Hz, and last for 1.5 sec.

$$x(t) = A \cos(\alpha t^2 + \beta t + \varphi)$$

$$\Rightarrow \alpha = ?$$

$$\Rightarrow \beta = ?$$

$$\Rightarrow \varphi = ?$$

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READING ASSIGNMENTS

- This Lecture:
 - Fourier Series in Ch 3, Sects 3-4, 3-5 & 3-6**
 - Replaces pp. 62-66 in Ch 3 in DSP First
 - Notation: \mathbf{a}_k for Fourier Series
- Other Reading:
 - Next Lecture: More Fourier Series

LECTURE OBJECTIVES

- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi k/T_0)t} dt$$

- ANALYSIS** via Fourier Series
 - For **PERIODIC** signals: $\mathbf{x}(t+T_0) = \mathbf{x}(t)$
 - Later: spectrum from the Fourier Series

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HISTORY

- Jean Baptiste Joseph Fourier
 - 1807 thesis (memoir)
 - On the Propagation of Heat in Solid Bodies
 - Heat !
 - Napoleonic era
- <http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html>



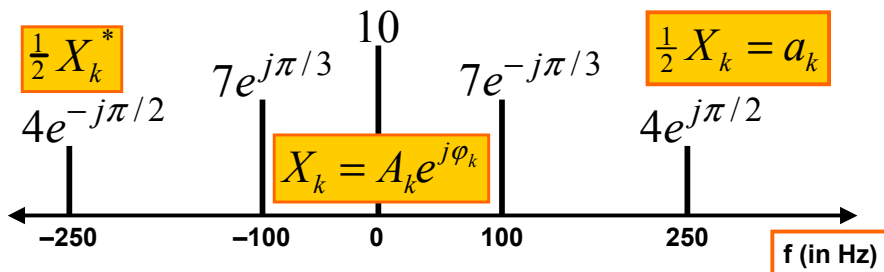
Joseph Fourier

lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = a_0 + \sum_{k=1}^N \{ a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \}$$

Harmonic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

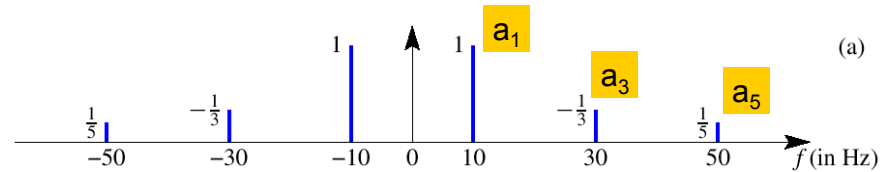
$$a_k = \frac{1}{2} X_k = \frac{1}{2} A_k e^{j\phi_k}$$

$$x(t) = A_0 + \sum_{k=1}^N A_k \cos(2\pi k f_0 t + \phi_k)$$

$$X_k = A_k e^{j\phi_k}$$

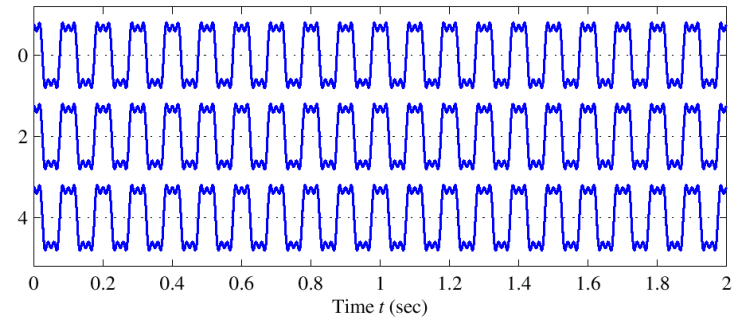
COMPLEX AMPLITUDE

Harmonic Signal (3 Freqs)



Sum of Cosine Waves with Harmonic Frequencies

T = 0.1



SYNTHESIS vs. ANALYSIS

- | | |
|---|---|
| <ul style="list-style-type: none"> ■ SYNTHESIS <ul style="list-style-type: none"> ■ Easy ■ Given (ω_k, A_k, ϕ_k) create $x(t)$ ■ Synthesis can be HARD <ul style="list-style-type: none"> ■ Synthesize Speech so that it sounds good | <ul style="list-style-type: none"> ■ ANALYSIS <ul style="list-style-type: none"> ■ Hard ■ Given $x(t)$, extract (ω_k, A_k, ϕ_k) ■ How many? ■ Need algorithm for computer |
|---|---|

STRATEGY: $x(t) \rightarrow a_k$

- ANALYSIS
 - Get representation from the signal
 - Works for **PERIODIC** Signals
- Fourier Series
 - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

INTEGRAL Property of exp(j)

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt} dt = \frac{T_0}{-j2\pi m} e^{-j(2\pi/T_0)mt} \Big|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m} (e^{-j2\pi m} - 1)$$

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt} dt = 0 \quad (m \neq 0) \quad \omega_0 = \frac{2\pi}{T_0}$$

ORTHOGONALITY of exp(j)

- PRODUCT of exp(+j) and exp(-j)

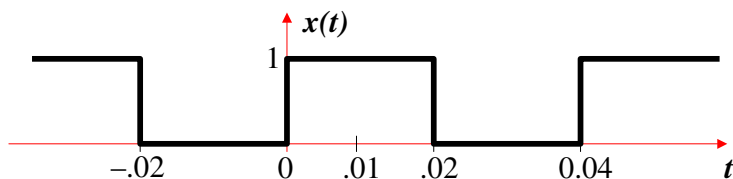
$$\frac{1}{T_0} \int_0^{T_0} e^{j(2\pi/T_0)\ell t} e^{-j(2\pi/T_0)kt} dt = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$$

$$\frac{1}{T_0} \int_0^{T_0} e^{j(2\pi/T_0)(\ell-k)t} dt$$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2}T_0 \\ 0 & \frac{1}{2}T_0 \leq t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



FS for a SQUARE WAVE {a_k}

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k \neq 0)$$

$$a_k = \frac{1}{.04} \int_0^{.02} 1 e^{-j(2\pi/.04)kt} dt = \frac{1}{.04(-j2\pi k/.04)} e^{-j(2\pi/.04)kt} \Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)} (e^{-j(\pi)k} - 1) = \frac{1 - (-1)^k}{j2\pi k}$$

DC Coefficient: a_0

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k = 0)$$

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt = \frac{1}{T_0} (\text{Area})$$

$$a_0 = \frac{1}{.04} \int_0^{.02} 1 dt = \frac{1}{.04} (.02 - 0) = \frac{1}{2}$$

Fourier Coefficients a_k

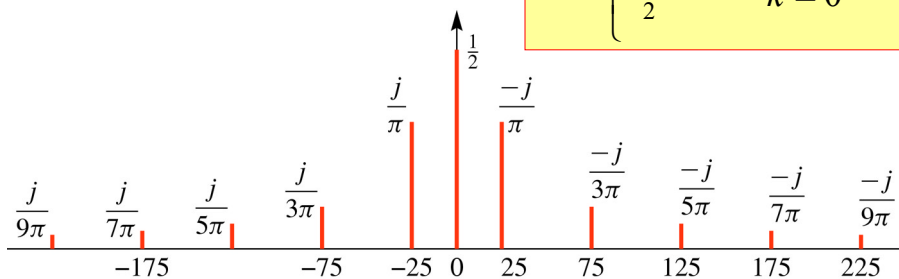
- a_k is a function of k
 - Complex Amplitude for k -th Harmonic
 - This one doesn't depend on the period, T_0

$$a_k = \frac{1 - (-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

$$\omega_0 = 2\pi / (0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \frac{-j}{\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$



Fourier Series Integral

- HOW do you determine a_k from $x(t)$?

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

Fundamental Frequency $f_0 = 1/T_0$

$a_{-k} = a_k^*$ when $x(t)$ is real

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt \quad (\text{DC component})$$