

Lecture 20
Continuous-Time Convolution
31-Mar-03

Tuesday Lab Change

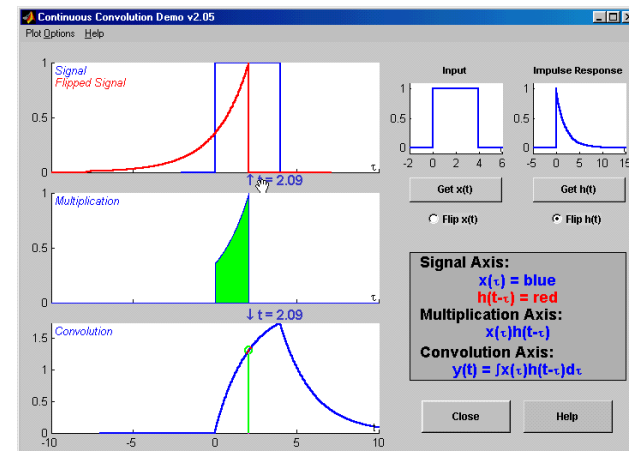
- No Lab on 1-April for the Tuesday sections
 - L02, L04 and L08
- Lab #11 will start on 8-April
 - Due on 15-April
- Lab #12 will start on 15-April
 - Due on 22-April
- Lab #10 should be turned in on 8-April

Info: Web-CT, Lab, HW

- Quiz #2: resolve grades by today
- Quiz #3 will be 11-April (Friday)
- HW #11 due this week
- Lab #10 on PeZ
- Lab #11 is about convolution
 - Cconvdemo

Lab #11 GUIs

- Continuous-time convolution demo:



Lecture

READING ASSIGNMENTS

- This Lecture:
 - Chapter 9, Sects. 9-6, 9-7, and 9-8
- Other Reading:
 - Recitation: Ch. 9, all
 - Next Lecture: Start reading Chapter 10

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LECTURE OBJECTIVES

- Review of C-T LTI systems
- **Evaluating convolutions**
 - **Examples**
 - **Impulses**
- LTI Systems
 - Stability and causality
 - Cascade and parallel connections

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Linear and Time-Invariant (LTI) Systems

- If a continuous-time system is both linear and time-invariant, then the output $y(t)$ is related to the input $x(t)$ by a **convolution integral**

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

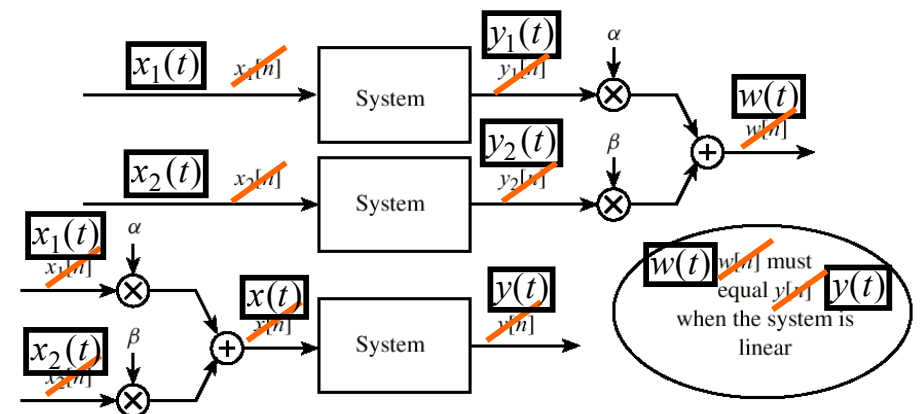
where $h(t)$ is the **impulse response** of the system.

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Testing for Linearity

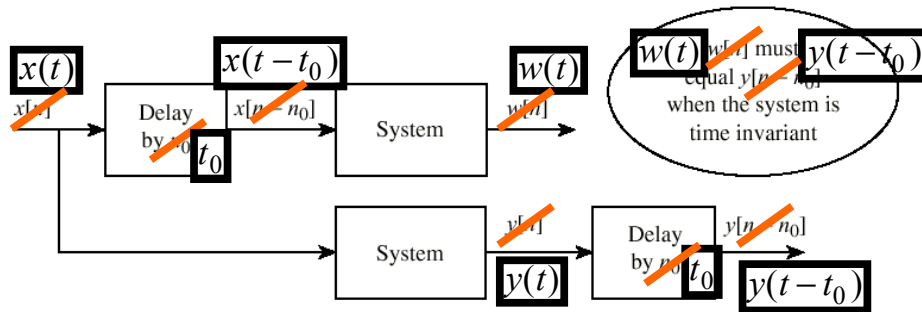


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Testing Time-Invariance



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Ideal Delay:

$$y(t) = x(t - t_d)$$

- Linear

$$ax_1(t - t_d) + bx_2(t - t_d) = ay_1(t) + by_2(t)$$

- and Time-Invariant

$$w(t) = x((t - t_d) - t_0)$$

$$y(t - t_0) = x((t - t_0) - t_d)$$

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Integrator:

$$y(t) = \int_{-\infty}^t x(\tau) d\tau$$

- Linear

$$\begin{aligned} \int_{-\infty}^t [ax_1(\tau) + bx_2(\tau)] d\tau &= \int_{-\infty}^t ax_1(\tau) d\tau + \int_{-\infty}^t bx_2(\tau) d\tau \\ &= ay_1(t) + by_2(t) \end{aligned}$$

- And Time-Invariant

$$w(t) = \int_{-\infty}^t x(\tau - t_0) d\tau \quad \text{Let } \sigma = \tau - t_0$$

$$\Rightarrow w(t) = \int_{-\infty}^{t-t_0} x(\sigma) d\sigma = y(t - t_0)$$

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Modulator:

$$y(t) = [A + x(t)] \cos \omega_c t$$

- Not** linear--obvious because

$$\begin{aligned} [A + ax_1(t) + bx_2(t)] &\neq \\ [A + ax_1(t)] + [A + bx_2(t)] \end{aligned}$$

- Not** time-invariant

$$w(t) = [A + x(t - t_0)] \cos \omega_c t \neq y(t - t_0)$$



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Linear and Time-Invariant (LTI) Systems

- If a continuous-time system is both linear and time-invariant, then the output $y(t)$ is related to the input $x(t)$ by a **convolution integral**

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau = x(t) * h(t)$$

where $h(t)$ is the **impulse response** of the system.

Convolution of Impulses, etc.

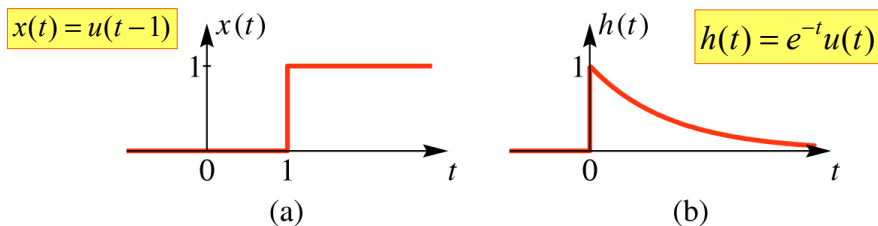
- Convolution of two impulses

$$\delta(t - t_1) * \delta(t - t_2) = \delta(t - t_1 - t_2)$$

- Convolution of step and shifted impulse

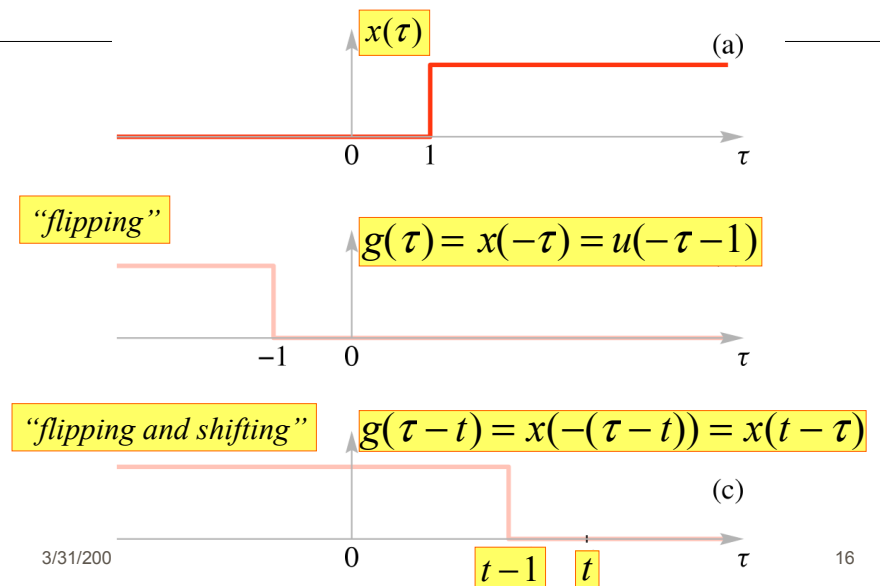
$$u(t) * \delta(t - t_0) = u(t - t_0)$$

Evaluating a Convolution

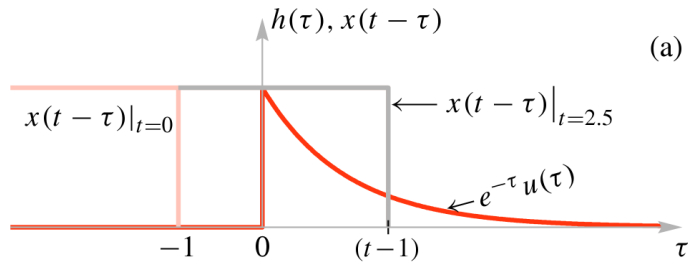


$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau = h(t) * x(t)$$

“Flipping and Shifting”



Evaluating the Integral



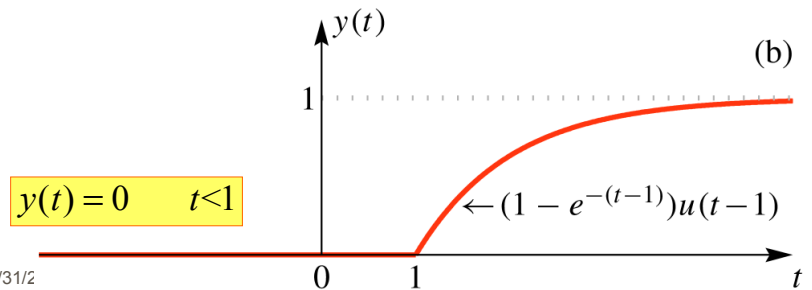
$$y(t) = \begin{cases} 0 & t-1 < 0 \\ \int_0^{t-1} e^{-\tau} d\tau & t-1 > 0 \end{cases}$$

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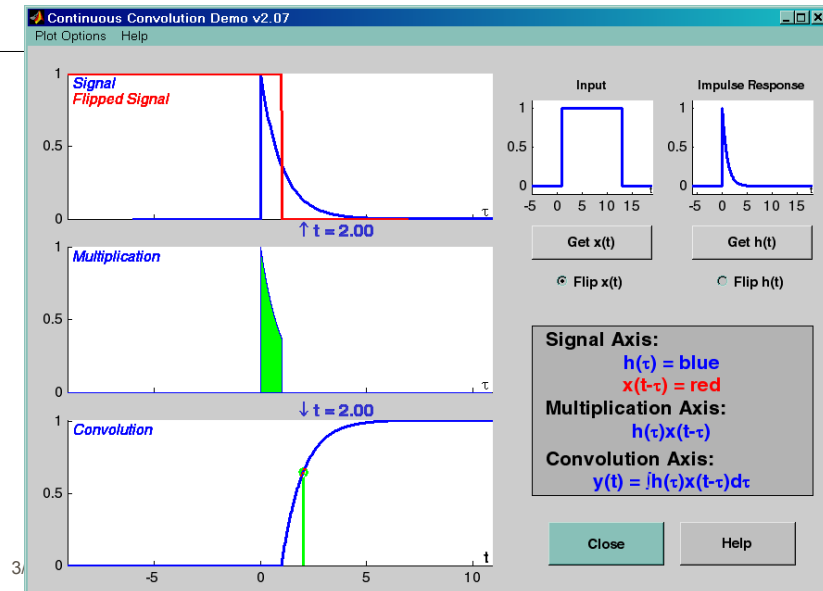
Solution

$$\begin{aligned} y(t) &= \int_0^{t-1} e^{-\tau} d\tau = -e^{-\tau} \Big|_0^{t-1} \\ &= 1 - e^{-(t-1)} \quad t \geq 1 \end{aligned}$$



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Convolution GUI



General Convolution Example

$$x(t) = e^{-at}u(t)$$

$$h(t) = e^{-bt}u(t), \quad b \neq a$$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t) \\ &= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)e^{-b(t-\tau)}u(t-\tau)d\tau = \begin{cases} e^{-bt} \int_0^t e^{-a\tau}e^{b\tau}d\tau & t > 0 \\ 0 & t < 0 \end{cases} \\ &= \begin{cases} \frac{e^{-at} - e^{-bt}}{-a+b} & t > 0 \\ 0 & t < 0 \end{cases} = \frac{e^{-at} - e^{-bt}}{b-a}u(t) \end{aligned}$$

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Special Case: $u(t)$

$$x(t) = e^{-at}u(t)$$

$$h(t) = u(t), \quad a \neq 0$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

$$= \frac{1}{a}(1 - e^{-at})u(t)$$

if $a = 2$

$$y(t) = \frac{1}{2}(1 - e^{-2t})u(t)$$

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Convolve Unit Steps

$$x(t) = u(t)$$

$$h(t) = u(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \begin{cases} \int_0^t 1d\tau & t > 0 \\ 0 & t < 0 \end{cases}$$

$$= \begin{cases} t & t > 0 \\ 0 & t < 0 \end{cases} = tu(t) \quad \text{Unit Ramp}$$

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Convolution is Commutative

$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

let $\sigma = t - \tau$ and $d\sigma = -d\tau$

$$h(t) * x(t) = - \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\sigma$$

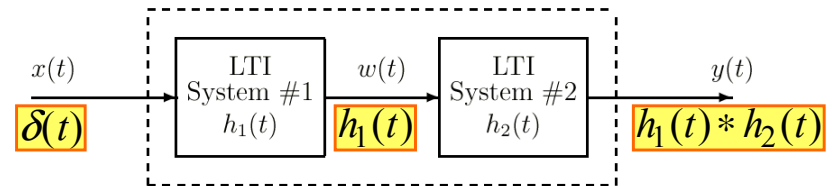
$$= \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\sigma = x(t) * h(t)$$

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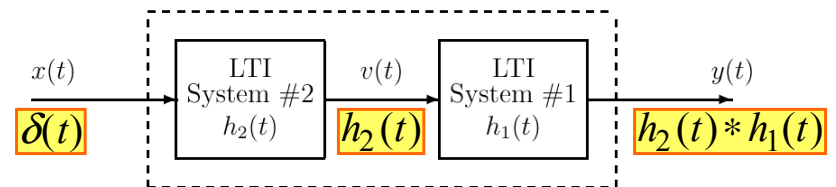
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Cascade of LTI Systems



$$h(t) = h_1(t) * h_2(t) = h_2(t) * h_1(t)$$



(b)

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Stability

- A system is stable if every bounded input produces a bounded output.
- A continuous-time *LTI system* is stable if and only if

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

Causal Systems

- A system is causal if and only if $y(t_0)$ depends only on $x(\tau)$ for $\tau \leq t_0$.
- An LTI system is causal if and only if

$$h(t) = 0 \text{ for } t < 0$$

Convolution is Linear

- Substitute $x(t) = ax_1(t) + bx_2(t)$

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} [ax_1(\tau) + bx_2(\tau)]h(t - \tau)d\tau \\ &= a \int_{-\infty}^{\infty} x_1(\tau)h(t - \tau)d\tau + b \int_{-\infty}^{\infty} x_2(\tau)h(t - \tau)d\tau \\ &= ay_1(t) + by_2(t) \end{aligned}$$

Therefore, convolution is linear.

Convolution is Time-Invariant

- Substitute $x(t - t_0)$

$$\begin{aligned} w(t) &= \int_{-\infty}^{\infty} h(\tau)x((t - \tau) - t_0)d\tau \\ &= \int_{-\infty}^{\infty} h(\tau)x((t - t_0) - \tau)d\tau \\ &= y(t - t_0) \end{aligned}$$