

Lecture 22
Introduction to the Fourier Transform
7-April-03

Info: Web-CT, Lab, HW

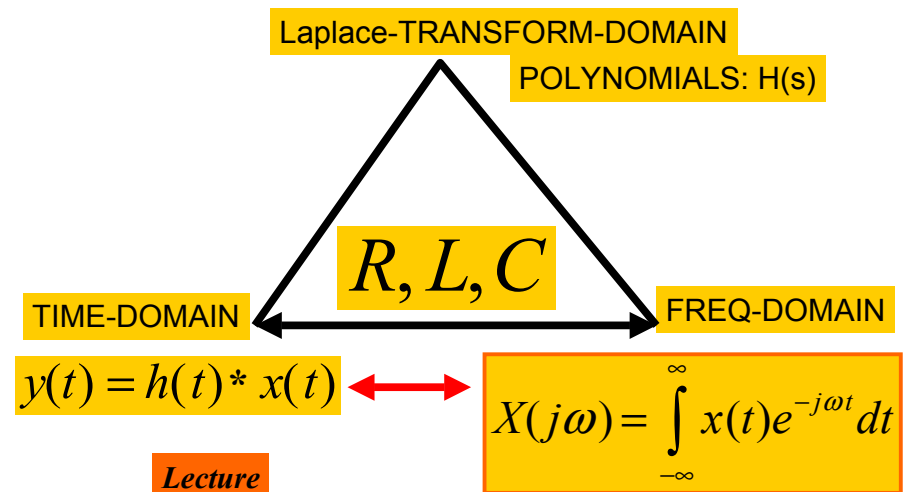
- Quiz #3 will be 11-April (Friday)
 - Coverage: HW #9, 10, 11, 12
 - Chapters 7, 8, and 9
 - One page hand-written notes (both sides)
 - Calculator OK
 - Review Session, 10-April, Thurs @ 7:30
- HW #12 due this week
- Lab #12 is the last lab
 - Fourier Series thru a BPF

Info: Lab #12

- Lab #12 uses FOURIER SERIES
 - {a_k} for Square Wave
- PreLab: Write MATLAB Functions:
 - ak4sqwave.m to evaluate {a_k}
 - and ak2sig.m to synthesize x_N(t)
- Also, Bandpass Filter
 - Use CLTIdemo to visualize
 - Errata:

$$H(j\omega) = \frac{j\omega}{\omega_c^2 - \omega^2 + j\omega}$$

THREE DOMAINS: ANALOG



READING ASSIGNMENTS

- This Lecture:
 - Chapter 11, Sects. 11-1 to 11-4
- Other Reading:
 - Recitation: Ch. 10
 - And Chapter 11, Sects. 11-1 to 11-4
 - Next Lecture: Chapter 11, Sects. 11-5, 11-6

LECTURE OBJECTIVES

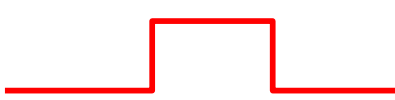
- Review
 - Frequency Response
 - Fourier Series
- Definition of **Fourier transform**

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

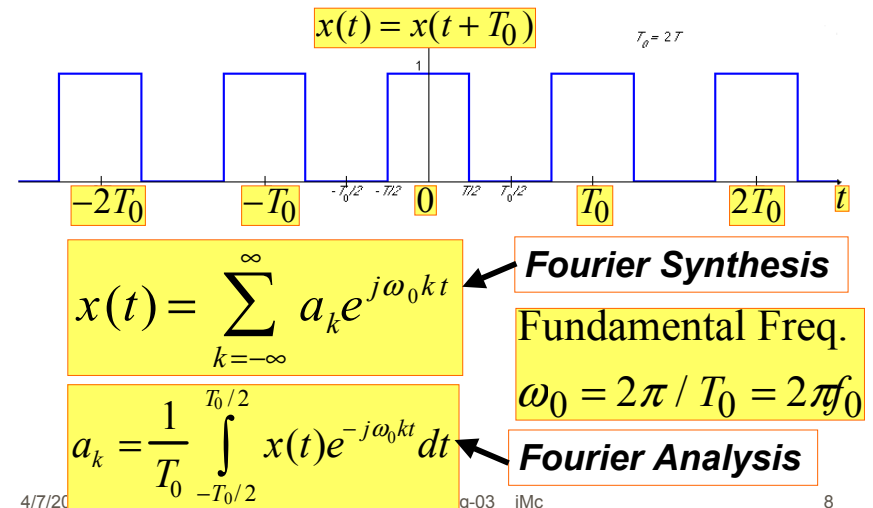
Relation to Fourier Series

- Examples of Fourier transform pairs

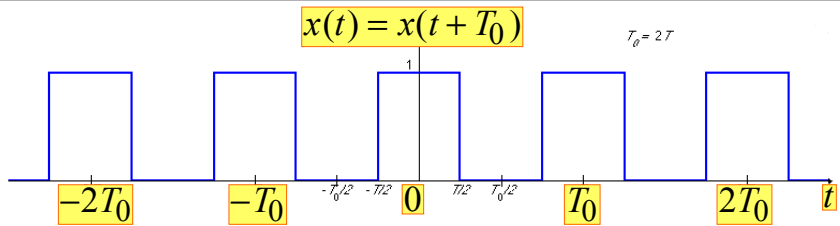
Everything = Sum of Sinusoids

- One Square Pulse = Sum of Sinusoids
 - ??????????????
 - Finite Length
 - Not Periodic
- 
- Limit of Square Wave as Period → infinity
 - Intuitive Argument

Fourier Series: Periodic $x(t)$



Square Wave Signal



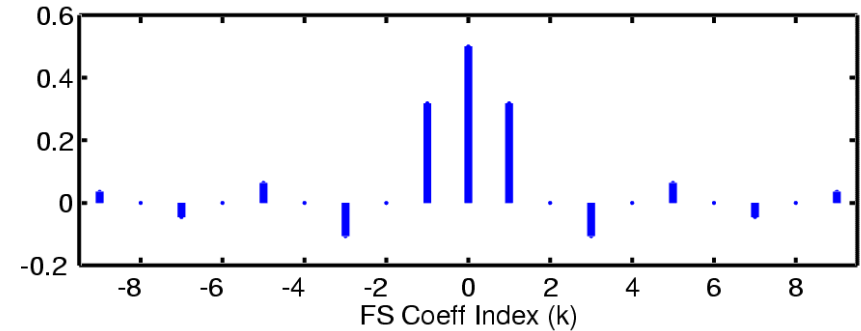
$$a_k = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} (1) e^{-j\omega_0 k t} dt$$

$$a_k = \frac{e^{-j\omega_0 k T_0} - 1}{-j\omega_0 k T_0} = \frac{e^{-j\pi k} - 1}{-j2\pi k} = \frac{\sin(\pi k / 2)}{\pi k}$$

Spectrum from Fourier Series

$$a_k = \frac{\sin(\pi k / 2)}{\pi k} = \begin{cases} \neq 0 & k = 0, \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \end{cases}$$

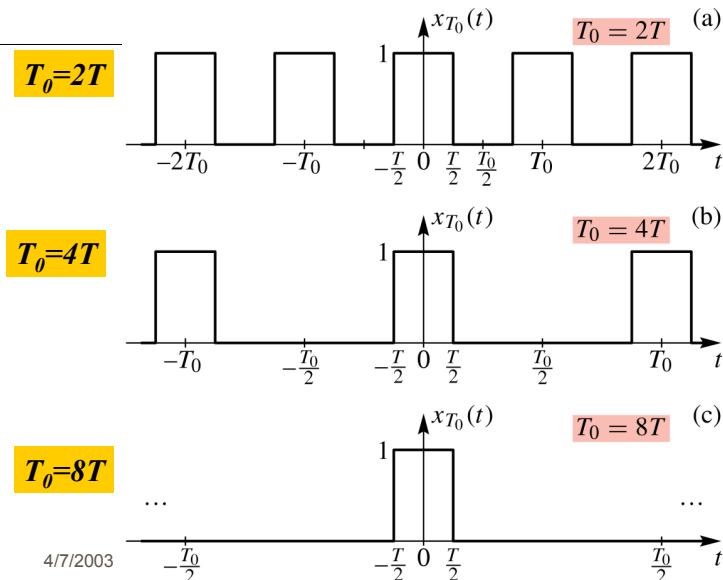
Fourier Series Coeffs for Square Wave



What if x(t) is not periodic?

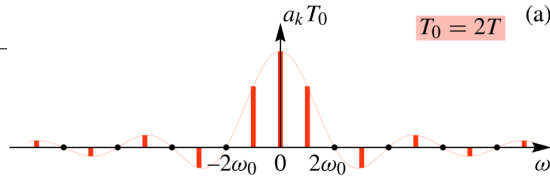
- Sum of Sinusoids?
 - Non-harmonically related sinusoids
 - Would not be periodic, but would probably be non-zero for all t.
- Fourier transform
 - gives a “sum” (actually an **integral**) that involves **ALL** frequencies
 - can represent signals that are identically zero for negative t. !!!!!!!!

Limiting Behavior of FS

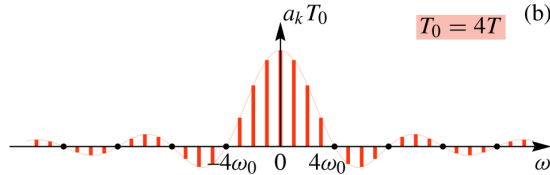


Limiting Behavior of Spectrum

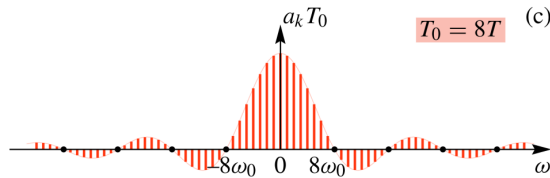
$T_0 = 2T$



$T_0 = 4T$



$T_0 = 8T$



Plot
($T_0 a_k$)

4/7/2003

13

FS in the LIMIT (long period)

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} (T_0 a_k) e^{j\omega_0 k t} \left(\frac{2\pi}{T_0} \right) \mapsto x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

$$\lim_{T_0 \rightarrow \infty} \frac{2\pi}{T_0} = d\omega$$

$$\lim_{T_0 \rightarrow \infty} \frac{2\pi}{T_0} k = \omega$$

$$\lim_{T_0 \rightarrow \infty} T_0 a_k = X(j\omega)$$

$$T_0 a_k = \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_0 k t} dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

4/7/2003

ECE-2025 Spring-03 jMc

14

Fourier Transform Defined

- For non-periodic signals

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

4/7/2003

ECE-2025 Spring-03 jMc

15

Example 1:

$$x(t) = e^{-at} u(t)$$

$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$X(j\omega) = -\frac{e^{-at} e^{-j\omega t}}{a+j\omega} \Big|_0^{\infty} = \frac{1}{a+j\omega} \quad a > 0$$

$$X(j\omega) = \frac{1}{a+j\omega}$$

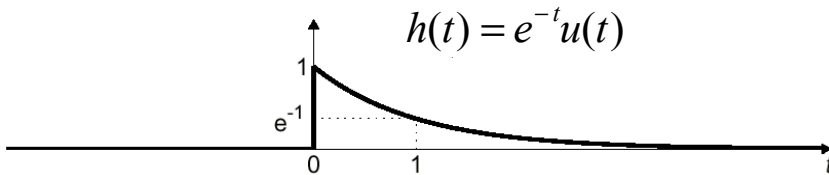
4/7/2003

ECE-2025 Spring-03 jMc

16

Frequency Response

- Fourier Transform of $h(t)$ is the Frequency Response



$$h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1 + j\omega}$$

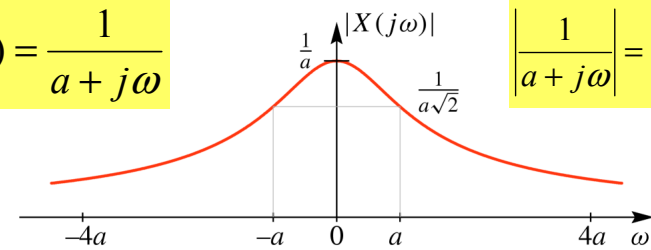
4/7/2003

ECE-2025 Spring-03 jMc

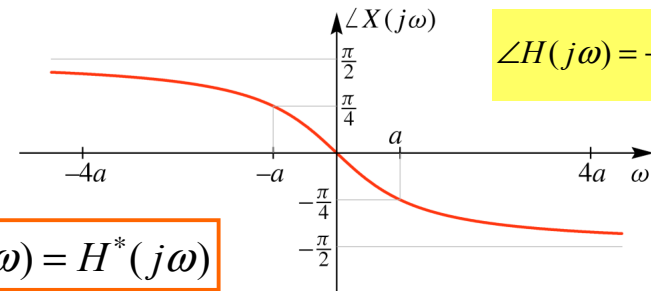
17

Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{a + j\omega}$$



$$\left| \frac{1}{a + j\omega} \right| = \frac{1}{\sqrt{a^2 + \omega^2}}$$



$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

$$H(-j\omega) = H^*(j\omega)$$

18

Example 2:

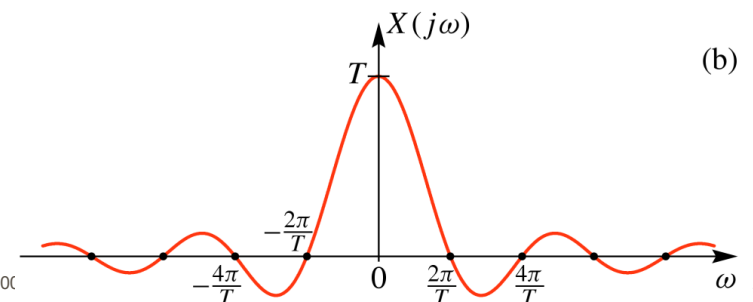
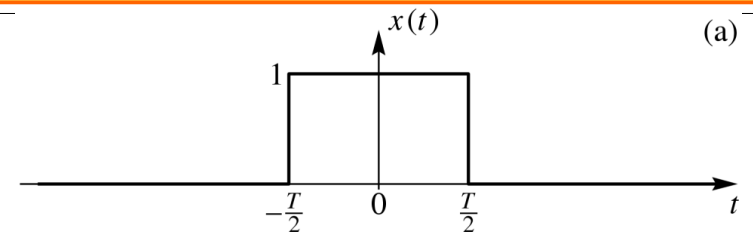
$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}$$

$$X(j\omega) = \int_{-T/2}^{T/2} (1)e^{-j\omega t} dt = \int_{-T/2}^{T/2} e^{-j\omega t} dt$$

$$X(j\omega) = \frac{e^{-j\omega t}}{-j\omega} \Big|_{-T/2}^{T/2} = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$X(j\omega) = \frac{\sin(\omega T / 2)}{(\omega / 2)}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T / 2)}{(\omega / 2)}$$



4/7/2003

ECE-2025 Spring-03 jMc

19

4/7/2003

20

Example 3:

$$X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega t} d\omega$$

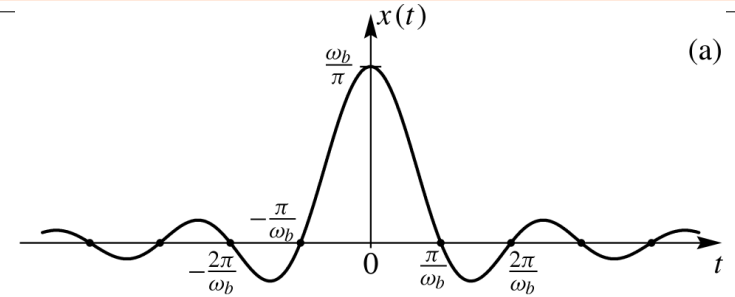
$$x(t) = \frac{1}{2\pi} \left. \frac{e^{j\omega t}}{jt} \right|_{-\omega_0}^{\omega_0} = \frac{1}{2\pi} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{jt}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)}$$

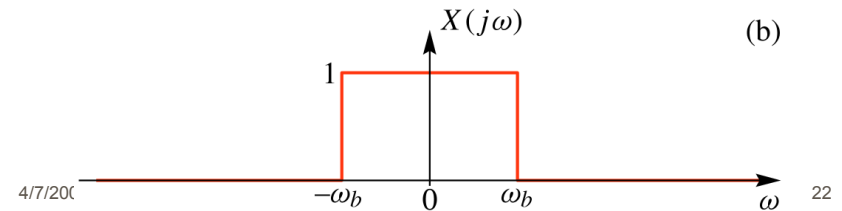
4/7/2003

21

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$



(a)



(b)

4/7/2003

22

Example 4:

$$x(t) = \delta(t - t_0)$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Shifting Property of the Impulse

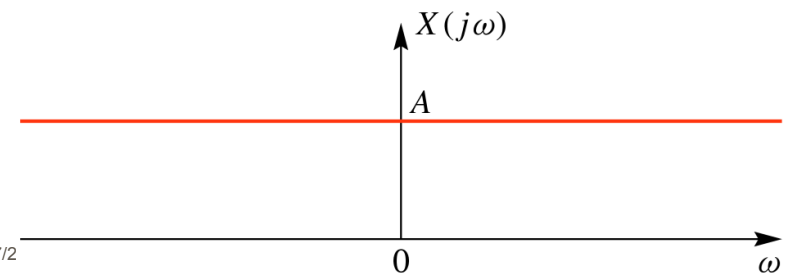
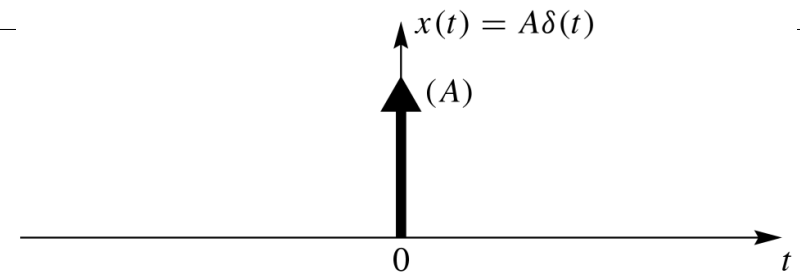
$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0}$$

4/7/2003

ECE-2025 Spring-03 jMc

23

$$x(t) = \delta(t) \Leftrightarrow X(j\omega) = 1$$



4/7/2003

24

Example 5: $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)$$

$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$

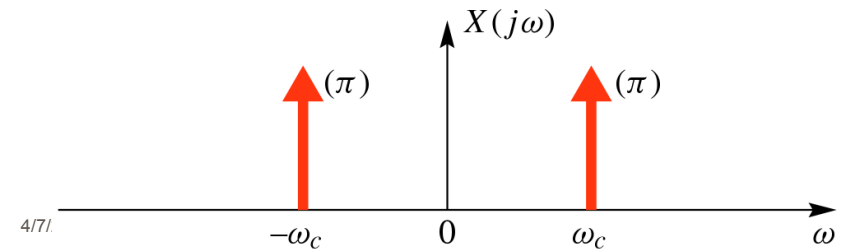
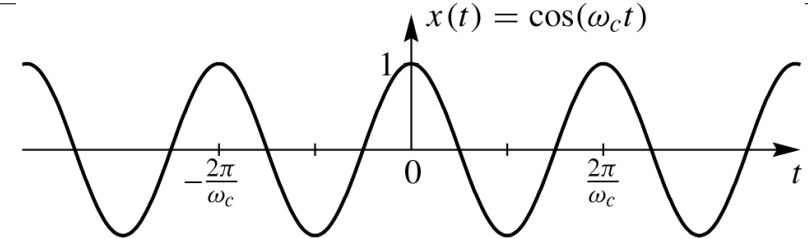
$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

4/7/20

25

$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



4/7/

Table of Fourier Transforms

$$x(t) = e^{-at} u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

4/7/20

27