ECE-2025 Spring-2003

Lecture 25
Sampling and Reconstruction (Fourier View)
21-Apr-03

Info: Web-CT, Lab, HW

- **Final Exam is Monday, 28-April**
  - Review Session on Sunday, 27-Apr
- Homework #14 due on **last** day, 25-April
- **Labs deadline to count for grade: 25-April**
- Grade Totals posted
  - Max is 70 points now
- Labs being held this week
  - One last on-line Post-Lab & 3 Surveys

Lectures

- A Lecture is the process in which the notes of the professor become the notes of the students …

  without passing through the minds of either.

LECTURE OBJECTIVES

- Sampling Theorem Revisited
  - GENERAL: in the FREQUENCY DOMAIN
  - Fourier transform of sampled signal
  - Reconstruction from samples
- Reading: Chap 12, Section 12-3
- Review of FT properties
  - Convolution ←→ multiplication
  - Frequency shifting
  - Review of AM

Table of FT Properties

$$x(t)*h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

Delay Property

$$x(t-t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

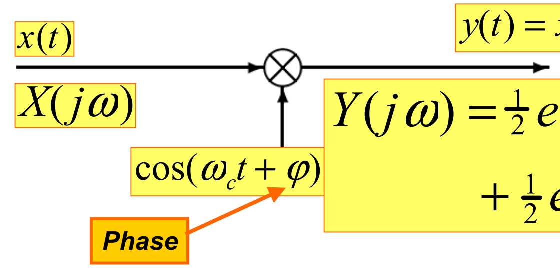
Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega-\omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\frac{\omega}{a}))$$

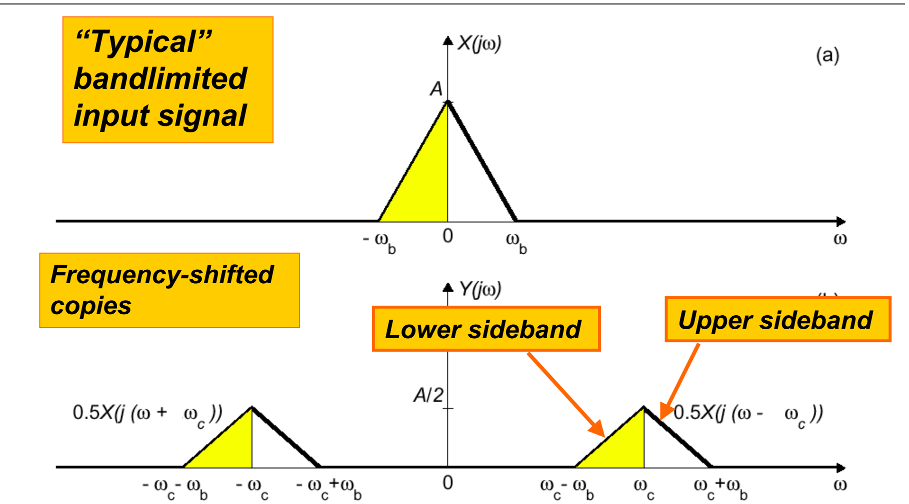
Amplitude Modulator



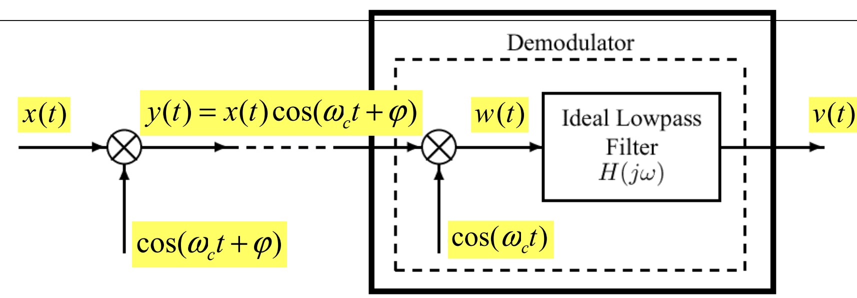
$$Y(j\omega) = \tfrac{1}{2} e^{j\varphi} X(j(\omega - \omega_c)) + \tfrac{1}{2} e^{-j\varphi} X(j(\omega + \omega_c))$$

- *x(t)* modulates the amplitude of the cosine wave. The result in the frequency-domain is two SHIFTED copies of $X(j\omega)$.

DSBAM: Frequency-Domain



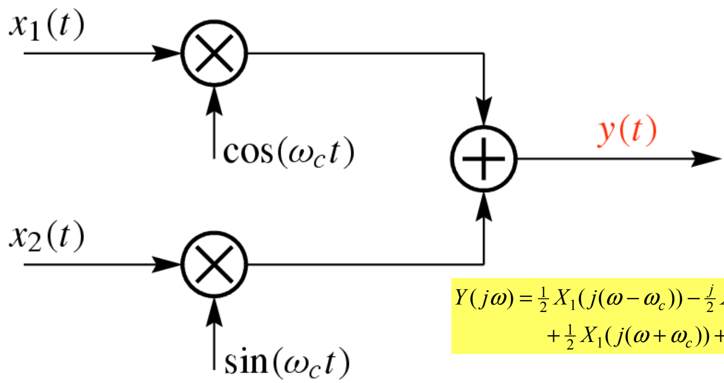
DSBAM Demod Phase Synch



$V(j\omega) = \frac{1}{2}\cos(\varphi)X(j\omega)$ what if $\varphi = \frac{1}{2}\pi$?

$$W(j\omega) = \tfrac{1}{4} e^{j\varphi} X(j\omega) + \tfrac{1}{4} e^{-j\varphi} X(j\omega) + \tfrac{1}{4} e^{j\varphi} X(j(\omega - 2\omega_c)) + \tfrac{1}{4} e^{-j\varphi} X(j(\omega + 2\omega_c))$$

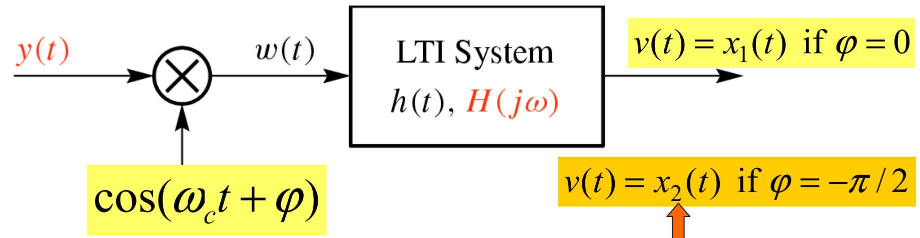
Quadrature Modulator



$$Y(j\omega) = \tfrac{1}{2}X_1(j(\omega-\omega_c)) - \tfrac{j}{2}X_2(j(\omega-\omega_c)) + \tfrac{1}{2}X_1(j(\omega+\omega_c)) + \tfrac{j}{2}X_2(j(\omega+\omega_c))$$

***TWO* signals on *ONE* channel: "out of phase"**
Can you "separate" them in the demodulator ?

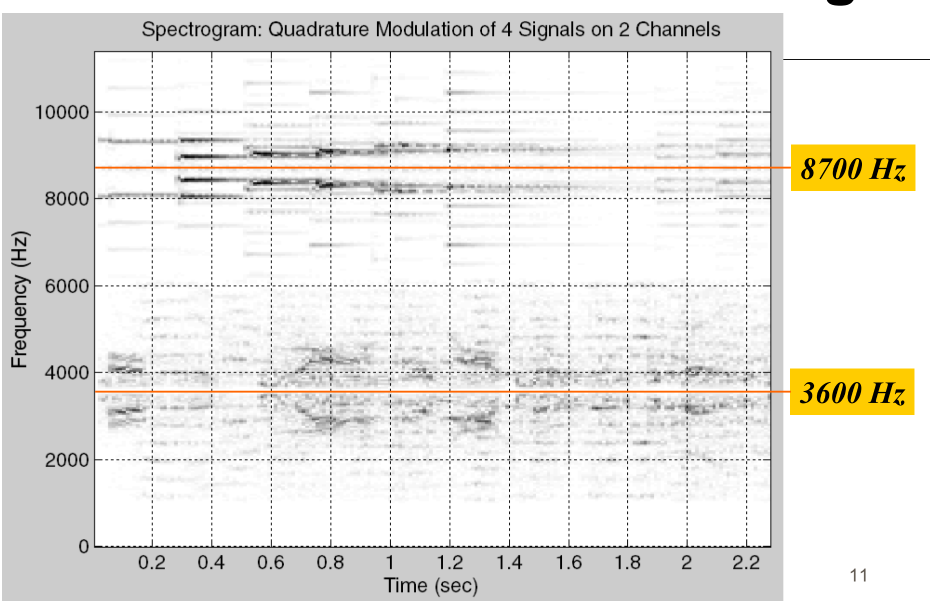
Demod: Quadrature System



$$Y(j\omega) = \tfrac{1}{2}X_1(j(\omega-\omega_c)) - \tfrac{j}{2}X_2(j(\omega-\omega_c)) + \tfrac{1}{2}X_1(j(\omega+\omega_c)) + \tfrac{j}{2}X_2(j(\omega+\omega_c))$$

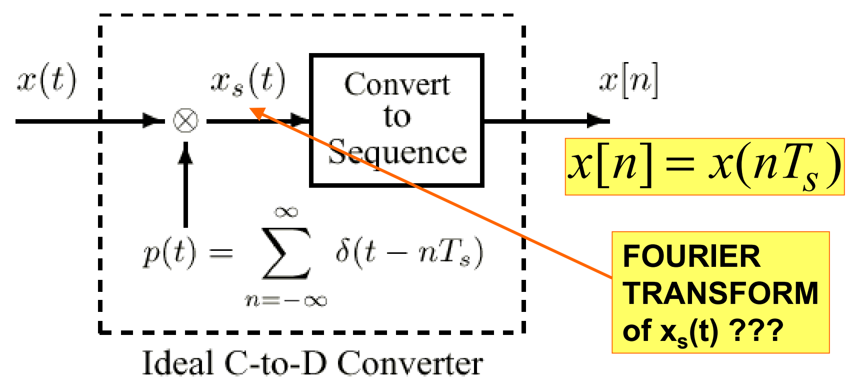
$$V(j\omega) = \tfrac{1}{4}e^{-j\varphi}X_1(j\omega) + \tfrac{1}{4}\underline{e^{-j\pi/2}e^{-j\varphi}}X_2(j\omega) + \tfrac{1}{4}e^{j\varphi}X_1(j\omega) + \tfrac{1}{4}\underline{e^{j\pi/2}e^{j\varphi}}X_2(j\omega)$$

Quadrature Modulation: 4 sigs

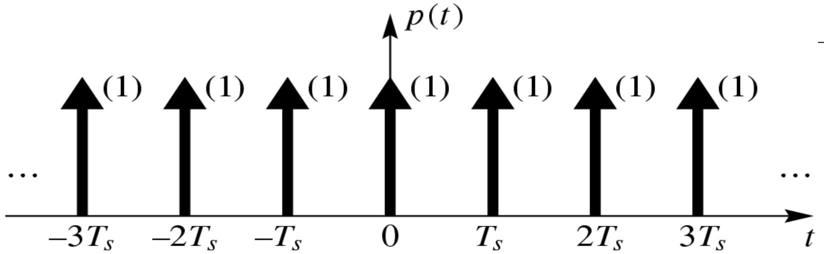


Ideal C-to-D Converter

- **Mathematical Model for A-to-D**



Periodic Impulse Train



$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_s)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_s t}$$

$$\omega_s=\frac{2\pi}{T_s}$$

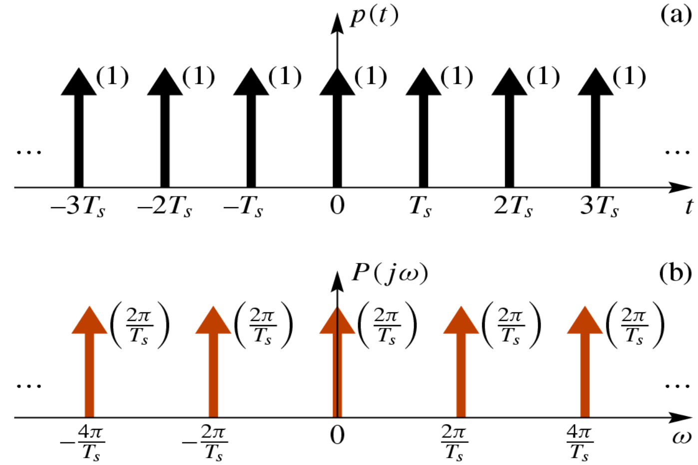
$$a_k=\frac{1}{T_s}\int_{-T_s/2}^{T_s/2}\delta(t)e^{-jk\omega_s t}dt=\frac{1}{T_s}$$

Fourier Series

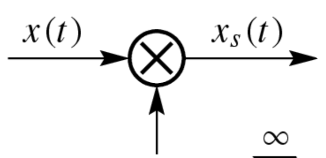
FT of Impulse Train

$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_s)\quad\leftrightarrow\quad P(j\omega)=\sum_{k=-\infty}^{\infty}\frac{2\pi}{T_s}\delta(\omega-k\omega_s)$$

$$\omega_s=\frac{2\pi}{T_s}$$



Impulse Train Sampling

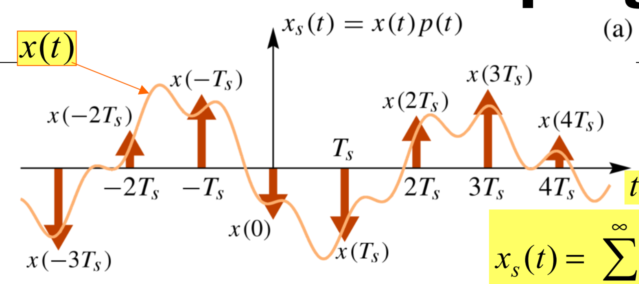


$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$$

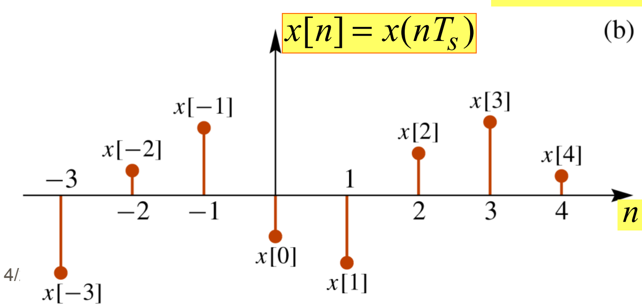
$$x_s(t)=x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s)=\sum_{n=-\infty}^{\infty}\underline{x(t)\delta(t-nT_s)}$$

$$x_s(t)=\sum_{n=-\infty}^{\infty}\underline{x(nT_s)\delta(t-nT_s)}$$

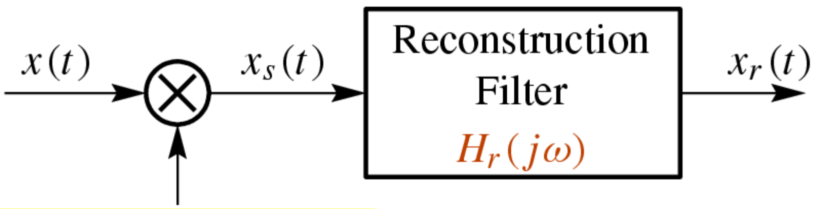
Illustration of Sampling



$$x_s(t)=\sum_{n=-\infty}^{\infty}x(nT_s)\delta(t-nT_s)$$



Sampling: Freq. Domain



$$p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

EXPECT FREQUENCY SHIFTING !!!

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

Frequency-Domain Analysis

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

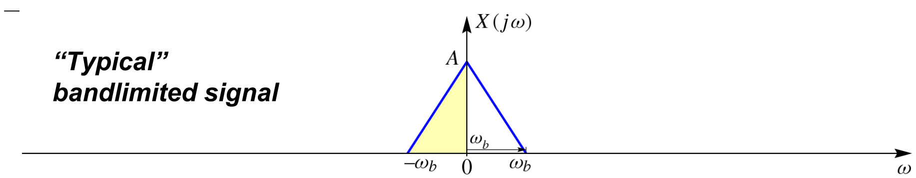
$$x_s(t) = x(t) \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} x(t) e^{jk\omega_s t}$$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

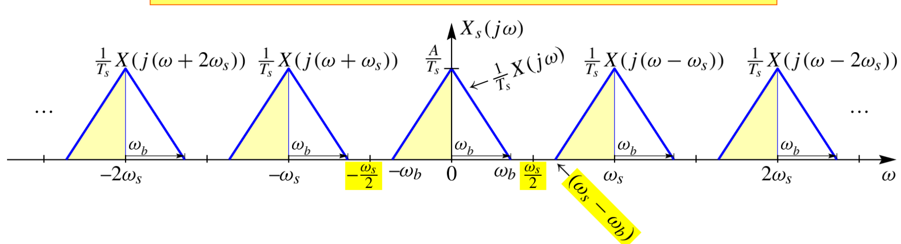
$$\omega_s = \frac{2\pi}{T_s}$$

Frequency-Domain Representation of Sampling

"Typical" bandlimited signal

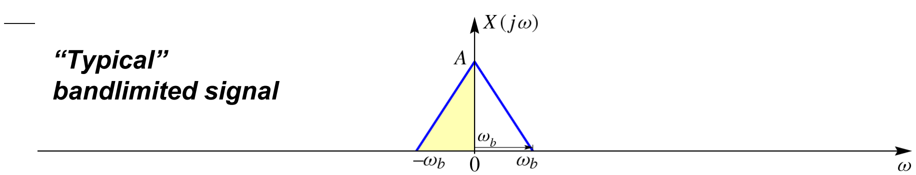


$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

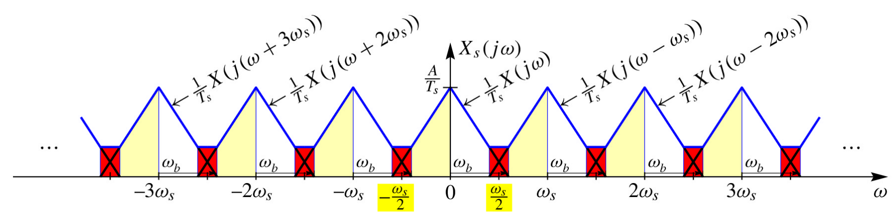


Aliasing Distortion

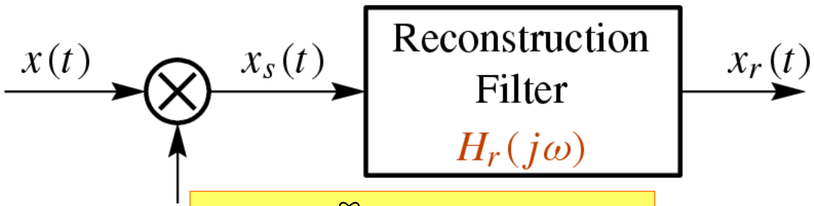
"Typical" bandlimited signal



- If $\omega_s < 2\omega_b$, the copies of $X(j\omega)$ overlap, and we have aliasing distortion.



Reconstruction of *x*(*t*)

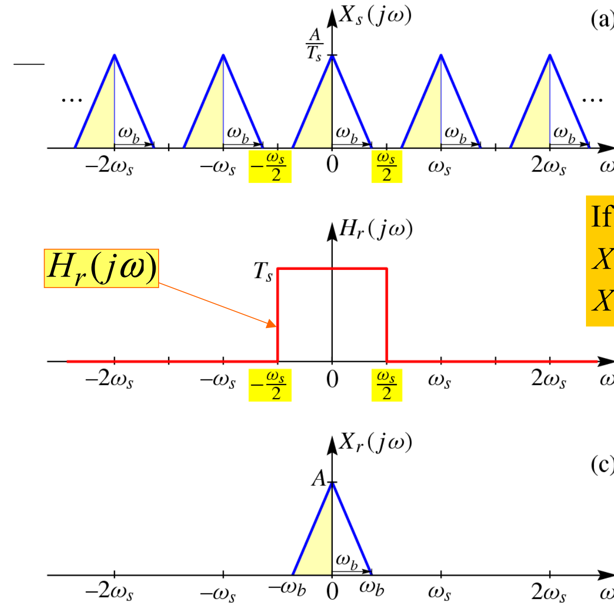


$$x_s(t)=\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t-nT_s)$$

$$X_s(j\omega)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega-k\omega_s))$$

$$X_r(j\omega)=H_r(j\omega)X_s(j\omega)$$

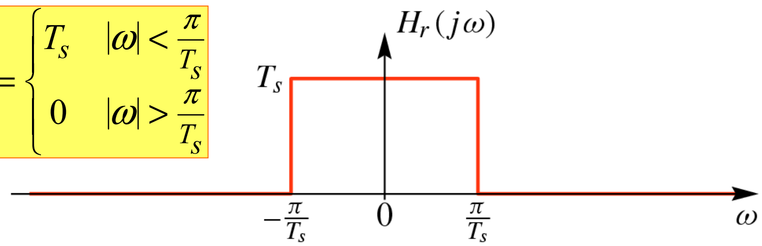
Reconstruction: Frequency-Domain



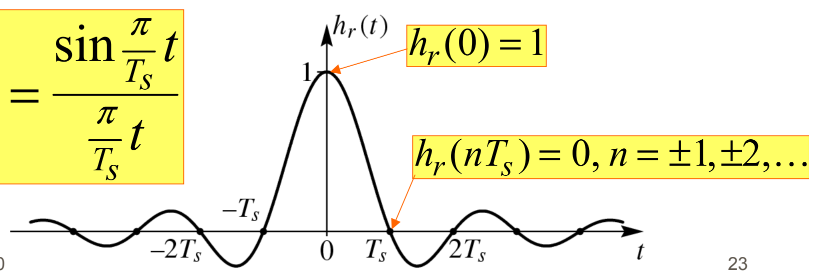
If $\omega_s > 2\omega_b$, the copies of $X(j\omega)$ do not overlap, so $X_r(j\omega)=H_r(j\omega)X_s(j\omega)$

Ideal Reconstruction Filter

$$H_r(j\omega)=\begin{cases} T_s & |\omega|<\frac{\pi}{T_s} \\ 0 & |\omega|>\frac{\pi}{T_s}\end{cases}$$



$$h_r(t)=\frac{\sin\frac{\pi}{T_s}t}{\frac{\pi}{T_s}t}$$



Signal Reconstruction

$$x_r(t)=h_r(t)*x_s(t)=h_r(t)*\sum_{n=-\infty}^{\infty}x(nT_s)\delta(t-nT_s)$$

$$x_r(t)=\sum_{n=-\infty}^{\infty}x(nT_s)h_r(t-nT_s)$$

$$x_r(t)=\sum_{n=-\infty}^{\infty}x(nT_s)\frac{\sin\frac{\pi}{T_s}(t-nT_s)}{\frac{\pi}{T_s}(t-nT_s)}$$

Ideal bandlimited interpolation formula

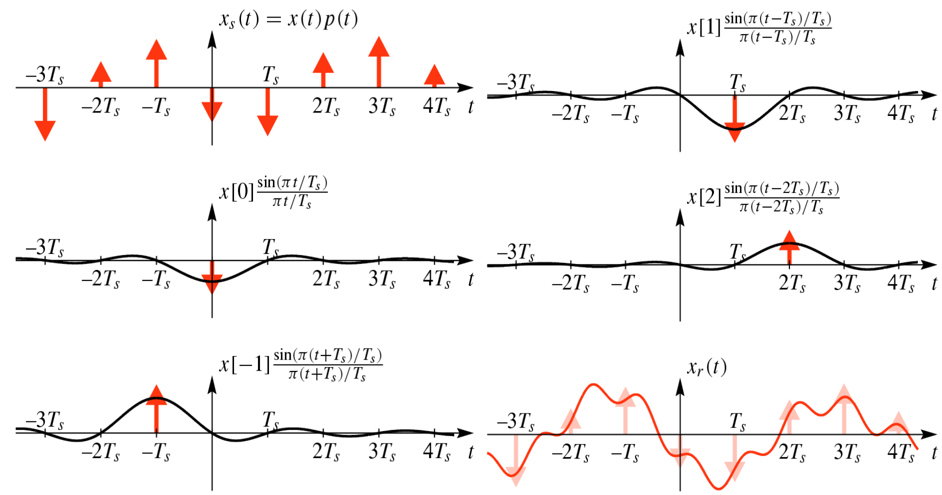
Shannon Sampling Theorem

- **"SINC" Interpolation** is the ideal
  - PERFECT RECONSTRUCTION
  - of BANDLIMITED SIGNALS

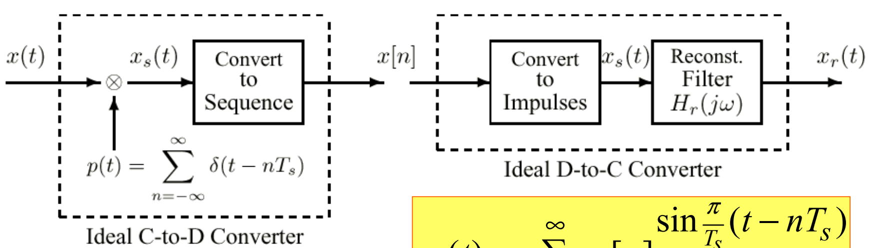
A signal $x(t)$ with bandlimited Fourier transform such that $X(j\omega) = 0$ for $|\omega| \geq \omega_b$ can be reconstructed exactly from samples taken with sampling rate $\omega_s = 2\pi/T_s \geq 2\omega_b$ using the following bandlimited interpolation formula:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin\left[\frac{\pi}{T_s}(t - nT_s)\right]}{\frac{\pi}{T_s}(t - nT_s)}.$$

Reconstruction in Time-Domain



Ideal C-to-D and D-to-C



$$x[n] = x(nT_s)$$

Ideal Sampler

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolator

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega)X_s(j\omega)$$