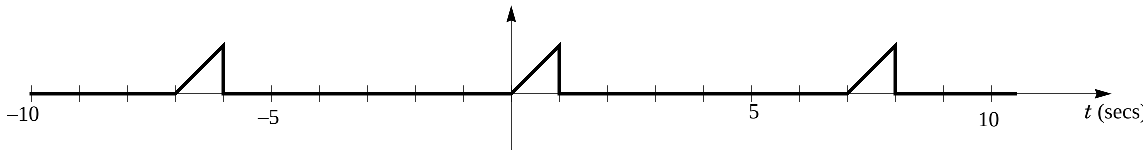
HOMEWORK 4 SOLUTIONS, ECE 2025 SPRING 2003

PROBLEM 4.1.

(a) Here is a sketch of $x(t)$, which has period $T = 7$:



(b) The DC value of $x(t)$ is the zero-th Fourier coefficient:

$$a_0 = \frac{1}{T}\int_0^T x(t)e^{-j(0)2\pi t/T}dt = \frac{1}{T}\int_0^T x(t)\,dt = \frac{1}{7}\int_0^7 x(t)\,dt = \frac{1}{7}\int_0^1 t\,dt = \frac{1}{7}\left[\frac{1}{2}t^2\right]_0^1 = \frac{1}{14}.$$

(c) The k-th Fourier coefficient is:

$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk2\pi t/T}dt = \frac{1}{7}\int_0^1 te^{-jk2\pi t/7}dt\,.$$

PROBLEM 4.2. A chirp that sweeps linearly from f_1 to f_2 as time goes from time $T_1 = 0$ to time T_2 is given by:

$$x(t) = \cos\left(2\pi f_1 t + \pi(f_2 - f_1)t^2/T_2\right)\,.$$

To verify that this formula is correct, recall that the instantaneous frequency (in Hertz) of $x(t) = \cos(\psi(t))$ is defined as:

$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\psi(t) = f_1 + (f_2 - f_1)t/T_2\,,$$

which is the formula for a straight line that ranges from f_1 to f_2 as time goes from 0 to T_2.

Plug in $f_1 = 8000$, $f_2 = 1000$, and $T_2 = 1.5$ to get the following formula for a chirp that sweeps from 8000 Hz down to 1000 Hz as time goes from zero to 1.5 seconds:

$$x(t) = \cos\left(2\pi(8000)t + \pi(1000 - 8000)t^2/1.5\right) = \cos\left(16000\pi t - (\tfrac{14000}{3})\pi t^2\right)\,.$$

PROBLEM 4.3. This problem pertains to $x(t) = 10\cos(1.2\pi t + 0.2\pi) + 15\cos(2.8\pi(t - 0.2)) - 5$.

(a) Yes, $x(t)$ is periodic. It is the sum of three terms: the first is periodic with frequency 0.6 Hz and period $1/(0.6\text{ Hz}) = 5/3$ seconds, the second is periodic with frequency 1.4 Hz and period $1/(1.4\text{ Hz}) = 5/7$ seconds, and the third term is a constant (which is trivially periodic with all periods). Over a span of 5 seconds, the first cosine completes three cycles, while the second completes seven cycles. Thus, the signal is periodic with period of $T = 5$ seconds. This is the fundamental period, since there is no smaller T such that $x(t + T) = x(t)$ for all t.

(b) The fundamental frequency is the inverse of the fundamental period: $f_0 = 1/T = 1/(5\text{ s.})$, or $f_0 = 0.2$ Hz. Observe that the 0.6 Hz and 1.4 Hz frequencies that contribute to $x(t)$ are both integer multiples of f_0.

(c) By inspection, the DC component to $x(t)$ is –5.

PROBLEM 4.4. Let f_n denote the frequency corresponding to the n-th key on the piano. For example, since the A above middle C is the 49-th key and its frequency is 440 Hz, we have $f_{49} = 440$ Hz.

(a) We are told that there are 12 tones in an octave, and that an octave doubles the frequency, so that $f_{n+12} = 2f_n$. We are also told that the ratio between successive frequencies is a constant, so that the ratio $\rho = f_{n+1}/f_n$ is independent of n. It follows that $f_{n+12} = \rho f_{n+11} = \rho^2 f_{n+10} = \rho^3 f_{n+9} = \ldots = \rho^{12} f_n$. Solving these two equations for ρ yields $\rho = 2^{1/12}$.

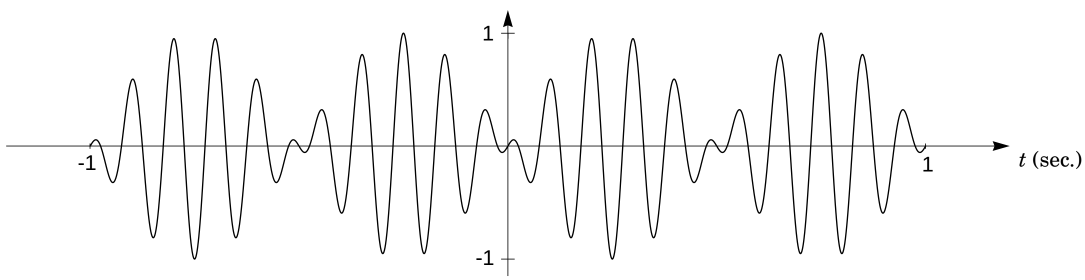
(b) Using the answer from part (c):
The MATLAB command `440*2.^(((40:52) - 49)/12)` yields:

n	40	41	42	43	44	45	46	47	48	49	50	51	52
f_n (Hz)	261.63	277.18	293.66	311.13	329.63	349.23	369.99	392.00	415.30	440	466.16	493.88	523.25

(c) The frequency of the n-th tone is $f_n = (440 \text{ Hz})\, 2^{(n-49)/12} = (25.96 \text{ Hz})\, 2^{n/12}$.

PROBLEM 4.5.

(a) Below is a sketch of $x(t) = \cos(20\pi t)\sin(2\pi t)$:



(b) To find its spectrum, express $\cos(\cdot)$ and $\sin(\cdot)$ in terms of complex exponentials:

$$\begin{aligned} x(t) &= \cos(20\pi t)\sin(2\pi t) \\ &= \frac{1}{2}(e^{j20\pi t} + e^{-j20\pi t})\frac{1}{2j}(e^{j2\pi t} - e^{-j2\pi t}) \\ &= \frac{1}{4j}\left(e^{j22\pi t} - e^{-j22\pi t} + e^{-j18\pi t} - e^{j18\pi t}\right). \end{aligned}$$

From this equation we get the following sketch for the spectrum: