ECE 2025

Solutions to Problem Set #7

Problem 7.1

(a) $y[n] = x[n+1] - x[n]$ (Forward Difference)

<u>Linearity:</u>

$$x[n] = \alpha_1 \cdot x_1[n] + \alpha_2 \cdot x_2[n]$$

$$x_1[n] \longmapsto y_1[n]$$

$$x_2[n] \longmapsto y_2[n]$$

$$\begin{aligned} y[n] &= \left(\alpha_1 \cdot x_1[n+1] + \alpha_2 \cdot x_2[n+1]\right) - \left(\alpha_1 x_1[n] + \alpha_2 \cdot x_2[n]\right) \\ &= \alpha_1 \left(x_1[n+1] - x_1[n]\right) + \alpha_2 \left(x_2[n+1] - x_2[n]\right) \\ &= \alpha_1 \cdot y_1[n] + \alpha_2 \cdot y_2[n] \end{aligned}$$

<u>$y[n]$ is linear</u>

<u>Time-invariance:</u>

Let $v[n] = x[n-n_0]$.

RHS: $v[n+1] - v[n] = x[n-n_0+1] - x[n-n_0]$

Since RHS is equal to $y[n-n_0] = x[n-n_0+1] - x[n-n_0]$,

<u>$y[n]$ is time-invariant.</u>

<u>Causal:</u>

$y[n]$ depends on $x[n+1]$. Therefore, <u>it is NOT causal.</u>

(b) $y[n] = |x[n]|^2$ (Magnitude Squared).

Linearity:

To prove $y[n]$ is nonlinear, it is enough to show one counter-example.

Let $x_1[n] = \delta[n]$ and $x_2[n] = -2\cdot\delta[n]$. Let's also assume $\alpha_1 = \alpha_2 = 1$.

$$
\begin{aligned}
y[n] &= |\alpha_1 \cdot x_1[n] + \alpha_2 \cdot x_2[n]|^2 \\
&= |\delta[n] - 2\cdot\delta[n]|^2 \\
&= (\delta[n])^2
\end{aligned}
$$

However,

$$
\begin{aligned}
y[n] &= \alpha_1 \cdot y_1[n] + \alpha_2 \cdot y_2[n] \\
&= 1\cdot|\delta[n]|^2 + 1\cdot|-2\cdot\delta[n]|^2 \\
&= 5\cdot(\delta[n])^2 \\
&\neq (\delta[n])^2
\end{aligned}
$$

Therefore, <u>$y[n]$ is NOT linear.</u>

Time-invariance:

Let $v[n] = x[n-n_0]$.

RHS: $|v[n]|^2 = |x[n-n_0]|^2$

Since RHS is equal to $y[n-n_0] = |x[n-n_0]|^2$, <u>$y[n]$ is time-invariant.</u>

Causal:

$y[n]$ depends only on the present value of the input.

Therefore, <u>$y[n]$ is causal.</u>

(c) $y[n] = x[-n]$ (Flip)

Linearity:

It is straightforward.

$$y[n] = \alpha_1 \cdot x_1[-n] + \alpha_2 \cdot x_2[-n]$$
$$= \alpha_1 \cdot y_1[n] + \alpha_2 \cdot y_2[n]$$

$y[n]$ is linear.

Time-invariance:

Let's first delay the input, and then flip it, we get

$v[n] = x[(-n)-n_0] = x[-n-n_0]$.

However, if we first flip the input sequence, and then delay it, we get

$y[n-n_0] = x[-(n-n_0)] = x[-n+n_0]$.

Therefore, $y[n]$ is NOT time-invariant.

Causal:

Let $n=-1$. Then, $y[-1] = x[1]$.

Therefore, $y[n]$ is NOT causal.

Problem 7.2

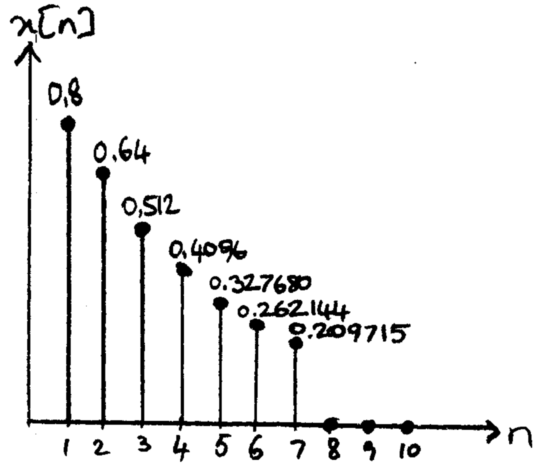
(a) $h[n] = \delta[n-1] - \beta \cdot \delta[n-2]$

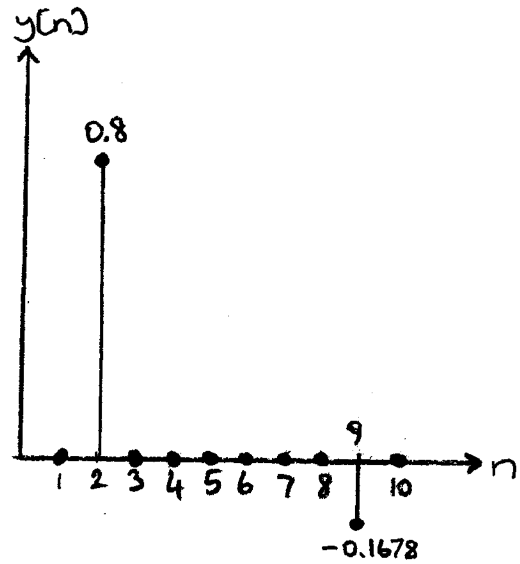
$y[n] = h[n] * x[n]$

n	0	1	2	3	4	5	6	7	8	9	10
$x[n]$	0	β	β^2	β^3	β^4	β^5	β^6	β^7	0	0	0
$h[n]$	0	1	$-\beta$								
$h[0]x[n]$	0	0	0	0	0	0	0	0	0	0	0
$h[1]x[n-1]$	0	0	β	β^2	β^3	β^4	β^5	β^6	β^7	0	0
$h[2]x[n-2]$	0	0	0	$-\beta^2$	$-\beta^3$	$-\beta^4$	$-\beta^5$	$-\beta^6$	$-\beta^7$	$-\beta^8$	0
$y[n]$	0	0	β	0	0	0	0	0	0	$-\beta^8$	0

As seen from the table, most of the values are zero.

(b)





Problem 7.3

(a) Substituting $x[n] = \delta[n]$ in $y_1[n]$, we get the impulse response sequence, $h_1[n]$.

$$h_1[n] = \sum_{k=1}^{7} \beta^k \cdot \delta[n-k]$$

$$= \beta \cdot \delta[n-1] + \beta^2 \cdot \delta[n-2] + \beta^3 \cdot \delta[n-3] + \ldots + \beta^7 \cdot \delta[n-7]$$

(b) $h[n] = h_1[n] * h_2[n]$

n	$n<0$	0	1	2	3	4	5	6	7	8	9	$n>9$
$h_1[n]$	0	0	β	β^2	β^3	β^4	β^5	β^6	β^7	0	0	0
$h_2[n]$	0	0	1	$-\beta$								
$h_2[0]h_1[n]$	0	0	0	0	0	0	0	0	0	0	0	0
$h_2[1]h_1[n-1]$	0	0	0	β	β^2	β^3	β^4	β^5	β^6	β^7	0	0
$h_2[2]h_1[n-2]$	0	0	0	0	$-\beta^2$	$-\beta^3$	$-\beta^4$	$-\beta^5$	$-\beta^6$	$-\beta^7$	$-\beta^8$	0
$h[n]$	0	0	0	β	0	0	0	0	0	0	$-\beta^8$	0

$$h[n] = \beta \cdot \delta[n-2] - \beta^8 \cdot \delta[n-9]$$

(c) Difference equation:

$$y[n] = h[n] * x[n]$$

$$= (\beta \cdot \delta[n-2] - \beta^8 \cdot \delta[n-9]) * x[n]$$

$$y[n] = \beta \cdot x[n-2] - \beta^8 \cdot x[n-9]$$

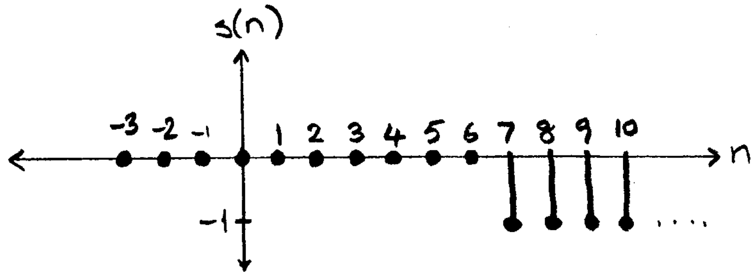
When $\beta = 0.8$

$$y[n] = 0.8 \cdot x[n-2] - 0.1678 \cdot x[n-9]$$

Problem 7.4

(a) $h[n] = \delta[n] - 3\cdot\delta[n-1] + 4\cdot\delta[n-2] - 3\cdot\delta[n-3] + \delta[n-4]$

(b)



Remember:

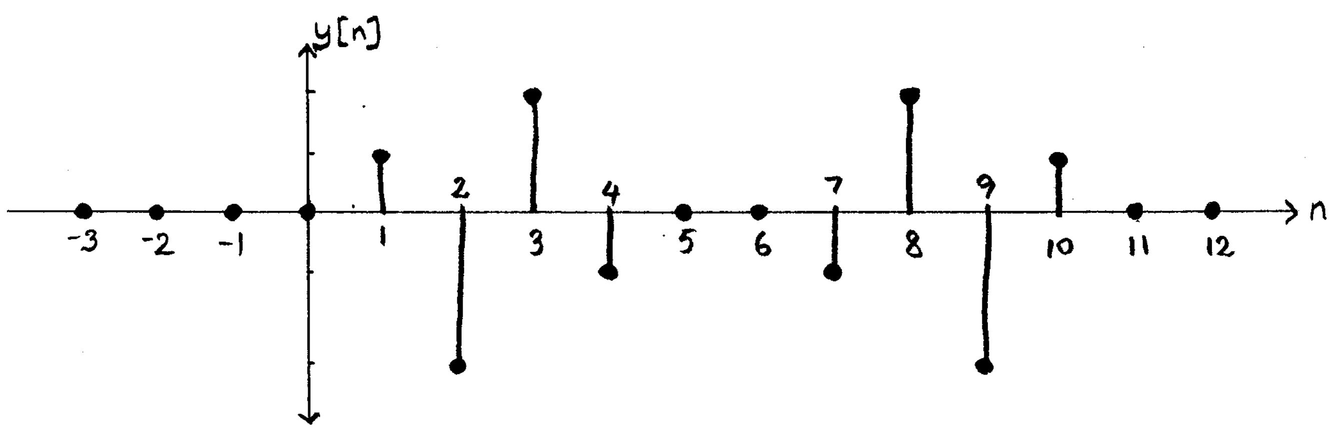
$$u[n] = \begin{cases} 0 & \text{for } n<0 \\ 1 & \text{for } n \geq 0 \end{cases}$$

(c) $y[n] = h[n] * x[n]$

$$x[n] = \begin{cases} 1 & n = 1,2,3,4,5,6 \\ 0 & \text{otherwise} \end{cases}$$

$$= \delta[n-1] + \delta[n-2] + \delta[n-3] + \delta[n-4] + \delta[n-5] + \delta[n-6]$$

n	$n<0$	0	1	2	3	4	5	6	7	8	9	10	$n>10$
$x[n]$	0	0	1	1	1	1	1	1	0	0	0	0	0
$h[n]$	0	1	-3	4	-3	1							
$h[0]\,x[n]$	0	0	1	1	1	1	1	1	0	0	0	0	0
$h[1]\,x[n-1]$	0	0	0	-3	-3	-3	-3	-3	-3	0	0	0	0
$h[2]\,x[n-2]$	0	0	0	0	4	4	4	4	4	4	0	0	0
$h[3]\,x[n-3]$	0	0	0	0	0	-3	-3	-3	-3	-3	-3	0	0
$h[4]\,x[n-4]$	0	0	0	0	0	0	1	1	1	1	1	1	0
$y[n]$	0	0	1	-2	2	-1	0	0	-1	2	-2	1	0



Problem 7.5

(a) $y[n] = \delta[n] + 2\cdot\delta[n-1] + 4\cdot\delta[n-2] + 3\cdot\delta[n-3] + 1.5\cdot\delta[n-4]$

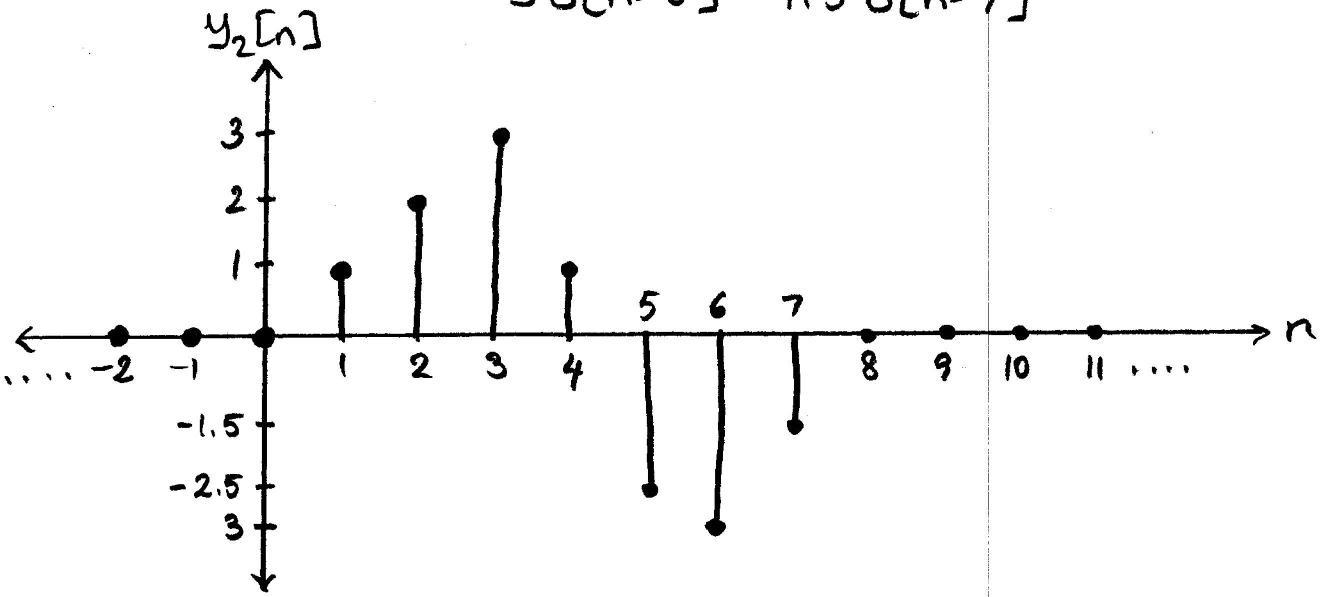
We must express $x_2[n]$ in terms of the known signal $x_1[n] = \delta[n+2]$.

$$x_2[n] = \delta[n+1] - \delta[n-1]$$
$$= x_1[n-1] - x_1[n-3]$$

Since the system is LTI,

$$x_1[n-1] \longrightarrow \delta[n-1] + 2\delta[n-2] + 4\delta[n-3] + 3\delta[n-4] + 1.5\delta[n-5]$$
$$-\ x_1[n-3] \longrightarrow \delta[n-3] + 2\delta[n-4] + 4\delta[n-5] + 3\delta[n-6] + 1.5\delta[n-7]$$

$$x_2[n] \longrightarrow \delta[n-1] + 2\delta[n-2] + 3\delta[n-3] + \delta[n-4] - 2.5\delta[n-5] - 3\delta[n-6] - 1.5\delta[n-7]$$



(b) A filter that uses only the present and past values of the input is called causal.

Let $x[n] = \delta[n]$. Then,

$$y[n] = \delta[n-2] + 2 \cdot \delta[n-3] + 4 \cdot \delta[n-4] + 3 \cdot \delta[n-5] + 1.5 \cdot \delta[n-6]$$

Since $y[n]$ depends only on the past values of the input, it is causal.

Problem 7.6

(a) $L = M+1$

(b) $$y_n = \sum_{k=0}^{9} b_k \cdot x[n-k]$$

$$= b_0 x[n] + b_1 x[n-1] + \dots + b_9 x[n-9]$$

The smallest value of n for which the first non-zero input value, $x[10]$, appears in the summation is $y[10]$.

$$y[10] = b_0 x[10] + b_1 x[9] + \dots + b_9 x[1]$$

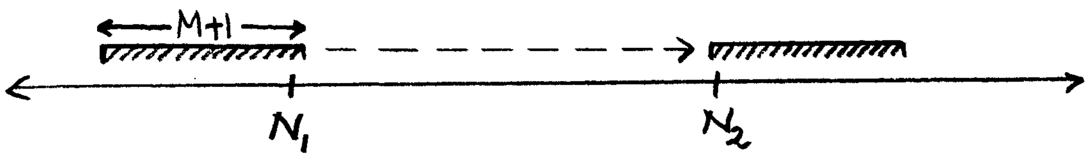
The largest value of n for which the last non-zero input value, $x[20]$, appears in the summation is $y[29]$.

$$y[29] = b_0 x[29] + b_1 x[28] + \dots + b_9 x[20]$$

Total length of the input sequence is 11.

(c) The length of the input sequence is $N_2 - N_1 + 1$.

(d) $y[n]$ is non-zero iff one of the samples from $x[N_1]$ to $x[N_2]$ is in the sum, i.e., we have a sliding window of length $M+1$ that must intersect the interval $[N_1, N_2]$.



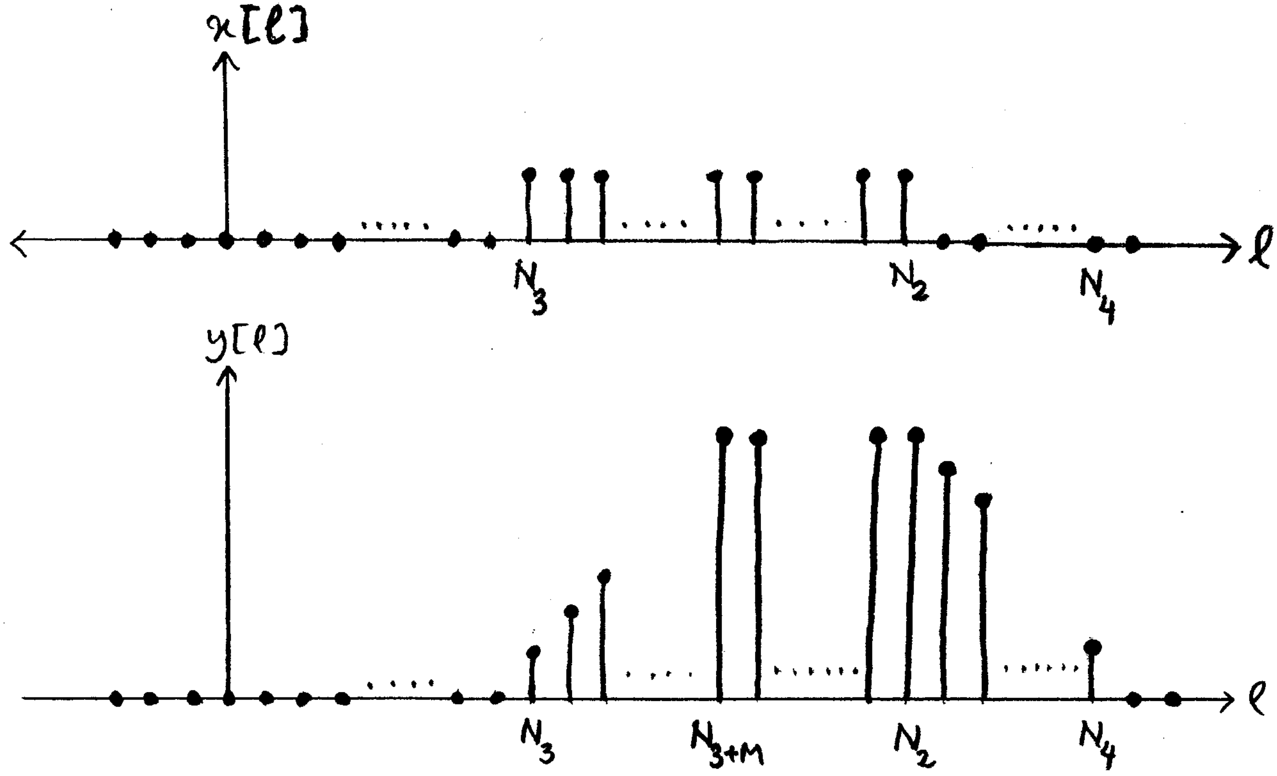
Since the largest index in the first case and the smallest index in the second case must be in the interval $[N_1, N_2]$, we conclude that

$$N_1 \le n \quad \text{and} \quad n - M \le N_2$$

Thus, $N_1 \le n \le N_2 + M$.

So, $N_3 = N_1$ and $N_4 = N_2 + M$.

(e)



The output is non-zero only for $N_3 \leq n \leq N_4$. Therefore, the length of the sequence is $N_4 - N_3 + 1$.