Solution to Problem Set 8

Problem 8.1:

(a)

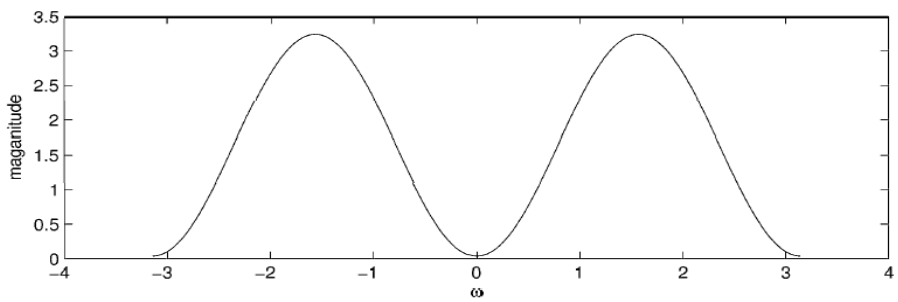
$$h_1[n] = \beta\delta[n] - \delta[n-2] \;\Rightarrow H_1\left(e^{j\hat{\omega}}\right) = \beta - e^{-j2\hat{\omega}} = e^{-j2\hat{\omega}}\left(\beta e^{j2\hat{\omega}} - 1\right)$$

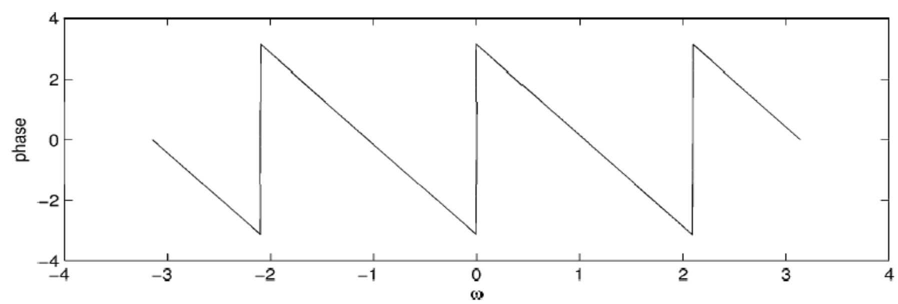
(b)

$$\begin{aligned} H_2\left(e^{j\hat{\omega}}\right) &= e^{-j\hat{\omega}} - \beta e^{-j3\hat{\omega}} = e^{-j\hat{\omega}}\left(1 - \beta e^{-j2\hat{\omega}}\right) \\ H\left(e^{j\hat{\omega}}\right) &= H_1\left(e^{j\hat{\omega}}\right)H_2\left(e^{j\hat{\omega}}\right) = e^{-j2\hat{\omega}}\left(\beta e^{j2\hat{\omega}} - 1\right)e^{-j\hat{\omega}}\left(1 - \beta e^{-j2\hat{\omega}}\right) \\ &= -e^{-j3\hat{\omega}}\left|1 - \beta e^{-j2\hat{\omega}}\right|^2 = \left(1 + \beta^2 - 2\beta\cos(2\hat{\omega})\right)e^{j(\pi - 3\hat{\omega})} \end{aligned}$$

(c)

$$\begin{aligned} \left|H\left(e^{j\hat{\omega}}\right)\right| &= 1 + \beta^2 - 2\beta\cos(2\hat{\omega}) \quad (\beta = 0.8) \\ &= 1.64 - 1.6\cos(2\hat{\omega}) \\ \phi(\hat{\omega}) &= \pi - 3\hat{\omega} \end{aligned}$$





(d)

$$\hat{\omega} = 0.3\pi, \quad \left|H\left(e^{j0.3\pi}\right)\right| = 1.64 - 1.6\cos(2 \bullet 0.3\pi) = 2.134 \quad \& \quad \phi(0.3\pi) = \pi - 3 \bullet 0.3\pi = 0.1\pi,$$

$$\hat{\omega} = \pi, \quad \left|H\left(e^{j\pi}\right)\right| = 1.64 - 1.6\cos(2 \bullet \pi) = 0.04 \quad \& \quad \phi(\pi) = \pi - 3 \bullet \pi = -2\pi \equiv 0,$$

$$\begin{aligned} y[n] &= 7 \bullet \left|H\left(e^{j0.3\pi}\right)\right|\cos(0.3\pi n - 0.4\pi + \phi(0.3\pi)) + \left|H\left(e^{j\pi}\right)\right|\cos(n\pi + \phi(\pi)) \qquad (\cos(n\pi) = (-1)^n) \\ &= 14.94\cos(0.3\pi n - 0.3\pi) + 0.04\cos(n\pi). \end{aligned}$$

Problem 8.2:

(a) Substituting $x_1[n] = 1 + (-1)^n$ into input/output relation, we have

$$\begin{aligned} y_1[n] &= -3G(1+(-1)^{n-1}) + 6G(1+(-1)^{n-2}) - 3G(1+(-1)^{n-3}) \\ &= (-3G(-1)^{-2} + 6G(-1)^{-3} - 3G(-1)^{-4})(-1)^{n+1} \\ &= -12G(-1)^{n+1}. \end{aligned}$$

Since $y_1[n] = 60(-1)^{n+1}$, $-12G = 60$ and $G = -5$.

If input is $x_2[n]$, which can be written as $x_2[n] = 25 - 10(1+(-1)^n) = 25 - 10x_1[n]$, then the output will be

$$\begin{aligned} y_1[n] &= -3G(25) + 6G(25) - 3G(25) - 10y_1[n] \\ &= -600(-1)^{n+1} = 600(-1)^n. \end{aligned}$$

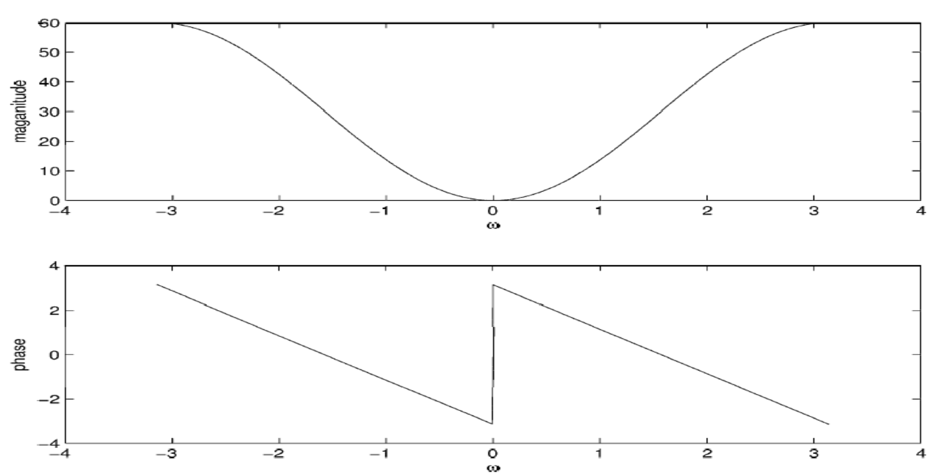
(b) From the input/output relation, we can find the impulse response of the system,

$$h[n] = -3G\delta[n-1] + 6G\delta[n-2] - 3G\delta[n-3].$$

Therefore,

$$\begin{aligned} H(e^{j\hat{\omega}}) &= -3Ge^{-j\hat{\omega}} + 6Ge^{-j2\hat{\omega}} - 3Ge^{-j3\hat{\omega}} \\ &= -3Ge^{-j2\hat{\omega}}(e^{j\hat{\omega}} - 2 + e^{-j\hat{\omega}}) \\ &= -3 \bullet (-5) \bullet e^{-j2\hat{\omega}} \bullet (e^{j\hat{\omega}/2} - e^{-j\hat{\omega}/2})^2 \\ &= 15e^{-j2\hat{\omega}}\left(2j\sin\left(\frac{\hat{\omega}}{2}\right)\right)^2 \\ &= 60\sin^2\left(\frac{\hat{\omega}}{2}\right)e^{j(\pi-2\hat{\omega})}. \end{aligned}$$

(c) From (b), $|H(e^{j\hat{\omega}})| = 60\sin^2\left(\frac{\hat{\omega}}{2}\right)$, and $\phi(\hat{\omega}) = \pi - 2\hat{\omega}$.



(d)

$$\hat{\omega} = 0, \quad \left|H\left(e^{j0}\right)\right| = 0 \qquad \& \qquad \phi(0) = 0,$$

$$\hat{\omega} = 0.5\pi, \quad \left|H\left(e^{j0.5\pi}\right)\right| = 60\sin^2\left(\frac{0.5\pi}{2}\right) = 30 \qquad \& \qquad \phi(0.5\pi) = \pi - 2 \bullet 0.5\pi = 0,$$

$$y[n] = 4 \bullet \left|H\left(e^{j0}\right)\right| + 8 \bullet \left|H\left(e^{j0.5\pi}\right)\right| \bullet \cos(0.5\pi n + \pi/2 + \phi(0.5\pi))$$

$$= 240\cos(0.5\pi n + \pi/2).$$

Problem 8.3:

$$H\left(e^{j\hat{\omega}}\right)=\left(1-e^{-j\hat{\omega}}\right)\left(1+e^{-j\pi/4}e^{-j\hat{\omega}}\right)\left(1+e^{j\pi/4}e^{-j\hat{\omega}}\right)$$
$$=\left(1-e^{-j\hat{\omega}}\right)\left(1+\sqrt{2}e^{-j\hat{\omega}}+e^{-j2\hat{\omega}}\right)$$
$$=1+\left(\sqrt{2}-1\right)e^{-j\hat{\omega}}-\left(\sqrt{2}-1\right)e^{-j2\hat{\omega}}-e^{-j3\hat{\omega}}$$

(a) Difference equation:

$$y[n]=x[n]+\left(\sqrt{2}-1\right)x[n-1]-\left(\sqrt{2}-1\right)x[n-2]-x[n-3].$$

(b) Impulse response:

$$h[n]=\delta[n]+\left(\sqrt{2}-1\right)\delta[n-1]-\left(\sqrt{2}-1\right)\delta[n-2]-\delta[n-3].$$

(c) The $\hat{\omega}$ that makes y[n] to be 0 for all n's satisfies

$$H\left(e^{j\hat{\omega}}\right)=0 \text{ or } \left(1-e^{-j\hat{\omega}}\right)\left(1+e^{-j\pi/4}e^{-j\hat{\omega}}\right)\left(1+e^{j\pi/4}e^{-j\hat{\omega}}\right)=0$$
$$\Downarrow$$
$$\hat{\omega}=0,\, 3\pi/4, \text{ or } -3\pi/4.$$

(d) The outputs corresponding to different inputs are as the following table

input	output
3	$3\bullet\left\|H\left(e^{j\omega 0}\right)\right\|=0$
$\delta[n-2]$	$h[n-2]=\delta[n-2]+\left(\sqrt{2}-1\right)\delta[n-3]-\left(\sqrt{2}-1\right)\delta[n-4]-\delta[n-5]$
$\cos(0.5\pi n+\pi/4)$	$\left\|H\left(e^{j0.5\pi}\right)\right\|\cos(0.5\pi n+\pi/4+\phi(0.5\pi))$ $=2\cos\left(0.5\pi n+\pi/4-\pi/4\right)$ $=2\cos(0.5\pi n)$

Therefore, overall output corresponding to x[n] will be the superposition of all three outputs in the above table, which is

$$y[n]=\delta[n-2]+\left(\sqrt{2}-1\right)\delta[n-3]-\left(\sqrt{2}-1\right)\delta[n-4]-\delta[n-5]+2\cos(0.5\pi n).$$

Problem 8.4:

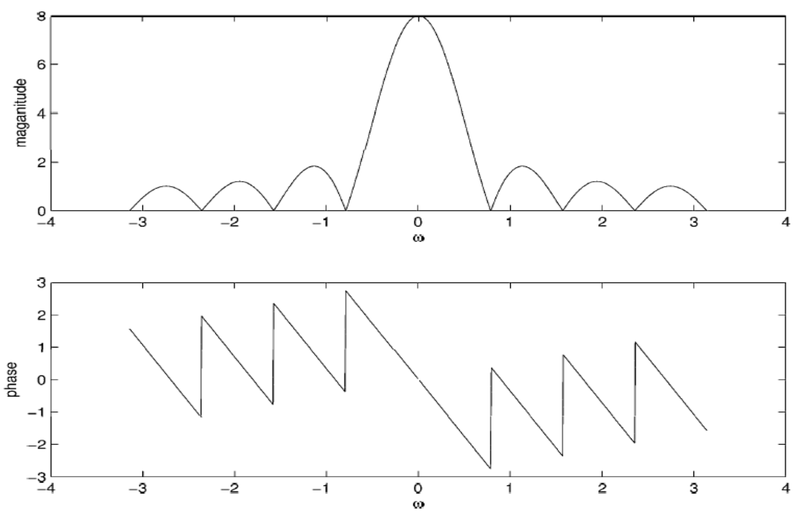
(a) $$H\left(e^{j\hat{\omega}}\right)=\sum_{k=0}^{7} e^{-jk\hat{\omega}}$$

(b)

$$H\left(e^{j\hat{\omega}}\right)=\sum_{k=0}^{7} e^{-jk\hat{\omega}}=\frac{1-e^{-j8\hat{\omega}}}{1-e^{-j\hat{\omega}}} \qquad \left(\sum_{k=0}^{N}\alpha^k=\frac{1-\alpha^{N+1}}{1-\alpha}\right)$$

$$=\frac{e^{j4\hat{\omega}}-e^{-j4\hat{\omega}}}{e^{j\hat{\omega}/2}-e^{-j\hat{\omega}/2}}\frac{e^{-j4\hat{\omega}}}{e^{-j\hat{\omega}/2}}=\frac{\sin(4\hat{\omega})}{\sin(\hat{\omega}/2)}e^{-j3.5\hat{\omega}}$$

(c) Magnitude and phase are as following:



(d) The necessary and sufficient condition for y[n] to be constant is

$$\left|H\left(e^{j\hat{\omega}}\right)\right|=\frac{\sin(4\hat{\omega}_0)}{\sin(\hat{\omega}_0/2)}e^{-j3.5\hat{\omega}_0}=0, \text{ or } \hat{\omega}_0=0$$

$$\Downarrow$$

$$4\hat{\omega}_0=n\pi, \text{for } n=1,...,7, \text{or } \hat{\omega}_0=0$$

$$\Downarrow$$

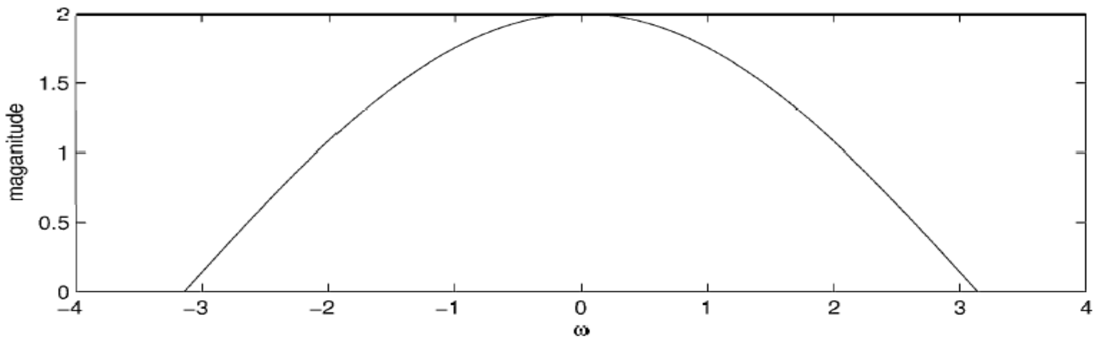
$$\hat{\omega}_0=n\pi/4, \text{for } n=0,1,...,7.$$

Problem 8.5:

(a) The frequency response magnitude of the system is

$$\left|H\left(e^{j\hat{\omega}}\right)\right| = \left|1+e^{-j\hat{\omega}}\right| = 2\left|\cos\left(\frac{\hat{\omega}}{2}\right)\right|,$$

which is plotted as the following.



(b) When $f_s = 300$ samples/sec, $\hat{\omega} = \frac{\omega}{f_s} = \frac{\omega}{300}$. To make $\left|H\left(e^{j\hat{\omega}}\right)\right| = 0$,

$$2\left|\cos\left(\frac{\hat{\omega}}{2}\right)\right| = 0$$

$$\Downarrow$$

$$\frac{\hat{\omega}}{2} = n\pi + \frac{\pi}{2} \text{ for all } n, \quad \text{or } \hat{\omega} = 2n\pi + \pi \text{ for all } n$$

$$\Downarrow$$

$$\omega = \hat{\omega} f_s = 600n\pi + 300\pi \text{ rad/sec}, \quad \text{or } f_s = 300n + 150 \text{ Hz for all } n.$$

(c) The sampled discrete signal is $x[n] = x(\frac{n}{f_s}) = 10 + 20\cos\left(\frac{\pi}{3}n\right)$. Direct evaluation of the frequency response yields that

$$\hat{\omega} = 0, \quad \left|H\left(e^{j0}\right)\right| = 2 \qquad \& \qquad \phi(0) = 0,$$

$$\hat{\omega} = \pi/3, \quad \left|H\left(e^{j\pi/3}\right)\right| = 2\left|\cos\left(\frac{\pi/3}{2}\right)\right| = \sqrt{3} \qquad \& \qquad \phi(\pi/3) = -\frac{\pi/3}{2} = -\frac{\pi}{6}.$$

Therefore, the LTI system output is

$$y[n] = 10 \bullet \left|H\left(e^{j0}\right)\right| + 20 \bullet \left|H\left(e^{j\pi/3}\right)\right| \bullet \cos(\frac{\pi}{3}n + \phi(\pi/3)) = 20 + 20\sqrt{3}\cos\left(\frac{\pi}{3}n - \frac{\pi}{6}\right).$$

And the output of D/C converter is $y(t) = y[t f_s] = 20 + 20\sqrt{3}\cos\left(100\pi t - \frac{\pi}{6}\right)$.