# Solutions HW #10

**10.1**

$$y[n] = y[n-3] + x[n]$$

a) Let $x[n] = \delta[n] \Rightarrow y[n] = h[n]$

| $n$ | $y[n]$ |
|---|---|
| 0 | $y[0] = y[-3] + 1 = 1$ |
| 1 | $y[1] = y[-2] + \cancel{x[1]}^{0} = 0$ |
| 2 | $y[2] = y[-1] + \cancel{x[2]}^{0} = 0$ |
| 3 | $y[3] = y[0] + x[3] = 1$ |
| 4 | $y[4] = 0$ |
| 5 | $y[5] = 0$ |
| 6 | $y[6] = 1$ |

$$y[n] = \boxed{\begin{aligned} h[n] &= 1 \qquad n = 0, 3, 6, \ldots \\ &= 0 \qquad \text{else} \end{aligned}}$$

b) Note: $h[n] = h[n-3] \Leftarrow$ period of $n=3$

c) $y[n] = -h[n-1] + 2h[n-4] - h[n-7]$

$$\begin{array}{rcccccccccccl}
= & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & & \ldots \\
+ & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & & \ldots \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & & & & \\
\hline
\end{array}$$

$$y[n] = [0 \;\; -1 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0 \;\; \ldots]$$

10.2 : $H(z) = \dfrac{1-z^{-1}}{1+0.9z^{-1}} = \underbrace{\dfrac{1}{1+0.9z^{-1}}}_{h[n] = (-0.9)^n u(n)} - \underbrace{\dfrac{z^{-1}}{1+0.9z^{-1}}}_{(-0.9)^{n-1} u(n-1)}$

$= h_2(n)$

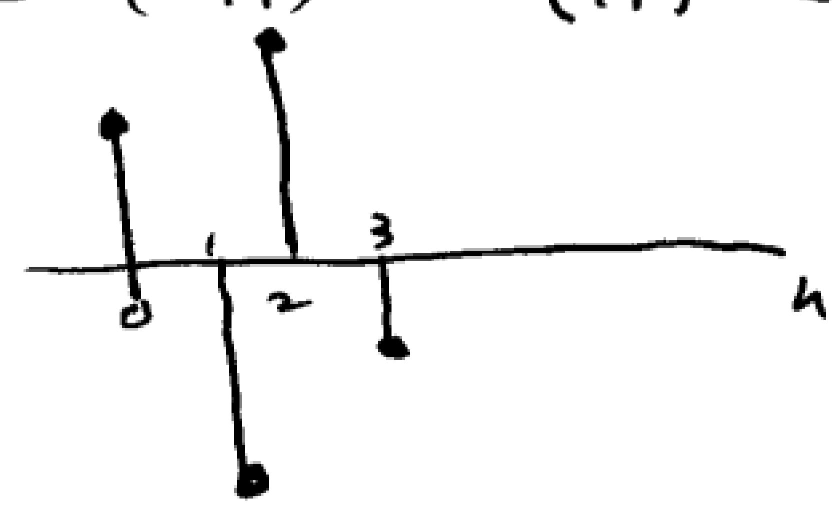
$$h[n] = (-0.9)^n u[n] - (-0.9)^{n-1} u[n-1]$$

$h(0) = 1$

$h(1) = -0.9 - 1 = -1.9$

$h(2) = (.9)^2 - (-0.9) = .81 + .9 = 1.71$

$h(3) = (-.9)^3 - (.9)^2 = -.729 - .81 = -1.54$




b) Note : $H(z) = \dfrac{1-z^{-1}}{1+0.9z^{-1}} = \dfrac{Y(z)}{X(z)}$

$\Rightarrow Y(z) = -0.9z^{-1}Y(z) + X(z) - z^{-1}X(z)$

$Y[n] = -0.9\,y[n-1] + x[n] - x[n-1]$

if $x[n] = u[n] \Rightarrow x[n] - x[n-1] = \delta(n)$

$\Rightarrow Y[n] = -0.9\,y[n-1] + \delta(n)$

| $n$ | $Y[n]$ |
|---|---|
| 0 | $Y[0] = -0.9\,y[n-1] + \delta[n] = 1$ |
| 1 | $Y[1] = -0.9(1) + 0 = -0.9$ |
| 2 | $Y[2] = (-0.9)(-0.9) + 0 = (-0.9)^2$ |

$$Y[n] = (-0.9)^n u[n]$$

c) To get a finite duration output, we need $Y(z)$ to have a <u>finite</u> # of terms.

e.g. if $Y(z) = 1 - z^{-1} \Rightarrow y[n] = [1, -1, 0, 0, \dots]$

finite duration.

How to get $Y(z) = 1 - z^{-1}$ ?

<u>recall</u> $Y(z) = H(z)\,X(z) = \dfrac{1 - z^{-1}}{1 + 0.9z^{-1}}\,X(z)$

so if $X(z) = 1 + 0.9z^{-1} \Rightarrow Y(z) = 1 - z^{-1}$

$\Rightarrow x[n] = \delta(n) + 0.9\,\delta(n-1)$

10.3

a) $x_a[n] = \left(-\frac{1}{2}\right)^n u[n-2]$

$m = n-2$

$$X_a(z) = \sum_{n=0}^{\infty} x_a[n] z^{-n} = \sum_{n=2}^{\infty} \left(-\frac{1}{2}\right)^n z^{-n} = \sum_{m=0} \left(-\frac{1}{2}\right)^{m+2} z^{-(m+2)}$$

$$= \frac{z^{-2}}{4} \sum_{m=0}^{\infty} \left(-\frac{1}{2}\right)^m z^{-m} = \frac{z^{-2}}{4} \sum_{m=0}^{\infty} \left(-\frac{1}{2} z^{-1}\right)^m$$

$$= \frac{z^{-2}}{4} \left( \frac{1}{1-\frac{1}{2} z^{-1}} \right)$$

b) $x_b[n] = 10(0.8)^n u[n] + 10(-0.8)^n u[n]$

$$X_b(z) = 10 \sum_{n=0}^{\infty} (0.8)^n z^{-n} + 10 \sum_{n=0}^{\infty} (-0.8)^n z^{-n}$$

$$= 10 \sum_{n=0}^{\infty} \left(\frac{4}{5} z^{-1}\right)^n + 10 \sum_{n=0}^{\infty} \left(-\frac{4}{5} z^{-1}\right)^n$$

$$= 10 \frac{1}{1-\frac{4}{5} z^{-1}} + 10 \frac{1}{1+\frac{4}{5} z^{-1}}$$

$$= 10 \left[ \frac{1+\frac{4}{5} z^{-1}}{\left(1-\frac{4}{5} z^{-1}\right)\left(1+\frac{4}{5} z^{-1}\right)} + \frac{1-\frac{4}{5} z^{-1}}{\left(1-\frac{4}{5} z^{-1}\right)\left(1+\frac{4}{5} z^{-1}\right)} \right]$$

$$= \frac{20}{1-\left(\frac{4}{5}\right)^2 z^{-2}}$$

10.3 c)

$$X_c(z) = \sum_{n=0}^{\infty} [\delta[n] - u[n]]\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \underbrace{\delta[n]}_{\substack{1 \; n=0 \\ 0 \; \text{else}}} z^{-n} - \sum_{n=0}^{\infty} u[n]\, z^{-n}$$

$$= 1 - \sum_{n=0}^{\infty} 1\, z^{-n}$$

$$= \frac{1 - z^{-1}}{1 - z^{-1}} - \frac{1}{1 - z^{-1}} = \frac{-z^{-1}}{1 - z^{-1}}$$

## PROBLEM 10.3*:

Determine the $z$-transforms of the following. *Express your answer as the ratio of polynomials in $z^{-1}$ by placing all terms over a common denominator.*

**Use the Z-transform pair:** $b(a)^n u[n] \rightarrow \dfrac{b}{1 - az^{-1}}$

(a) $x_a[n] = \left(-\frac{1}{2}\right)^n u[n-2]$

$$x_a[n] = \left(-\tfrac{1}{2}\right)^n u[n-2] = \left(-\tfrac{1}{2}\right)^2 \left(-\tfrac{1}{2}\right)^{n-2} u[n-2]$$

$$(-\tfrac{1}{2})^n u[n] \rightarrow \frac{1}{1 - (-\frac{1}{2})z^{-1}}$$

$$(-\tfrac{1}{2})^{n-2} u[n-2] \rightarrow \frac{z^{-2}}{1 - (-\frac{1}{2})z^{-1}} \qquad \textbf{(Delay Property)}$$

$$x_a[n] \rightarrow X_a(z) = \frac{\frac{1}{4}z^{-2}}{1 + \frac{1}{2}z^{-1}}$$

(b) $x_b[n] = 10(0.8)^n u[n] + 10(-0.8)^n u[n]$

$$10(0.8)^n u[n] \rightarrow \frac{10}{1 - 0.8z^{-1}}$$

$$10(-0.8)^n u[n] \rightarrow \frac{10}{1 - (-0.8)z^{-1}}$$

$$x_b[n] \rightarrow X_b(z) = \frac{10}{1 - 0.8z^{-1}} + \frac{10}{1 - (-0.8)z^{-1}}$$

$$= \frac{10(2) + 8z^{-1} - 8z^{-1}}{(1 - 0.8z^{-1})(1 + 0.8z^{-1})}$$

$$= \frac{20}{1 - 0.64z^{-2}}$$

(c) $x_c[n] = \delta[n] - u[n]$

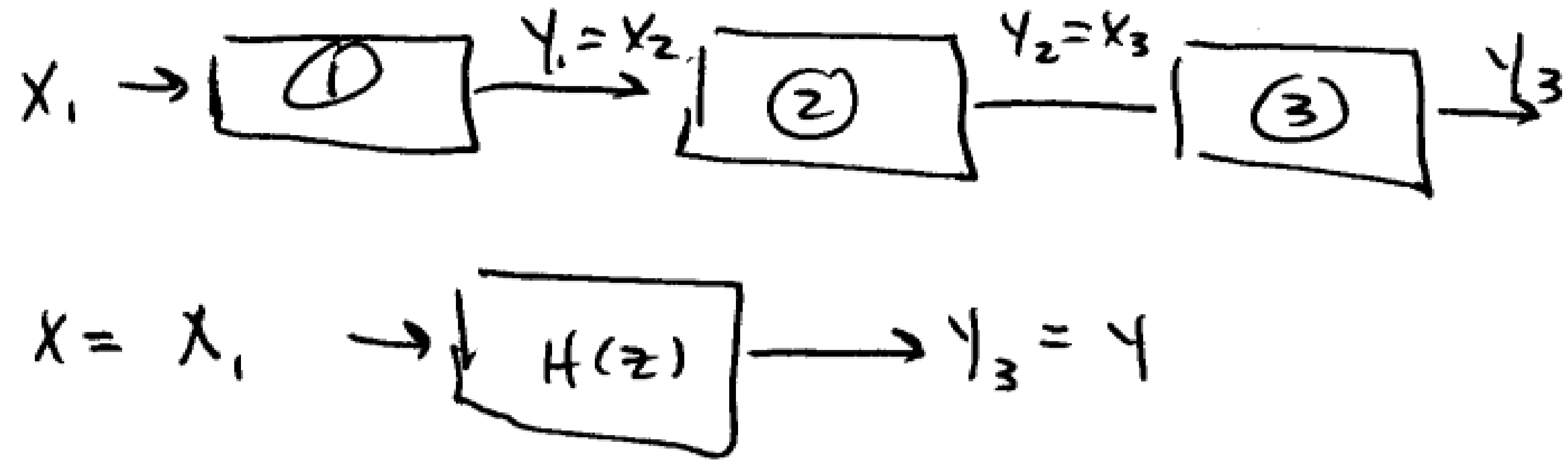
$$\delta[n] \rightarrow \Delta(z) = 1$$

$$u[n] \rightarrow U(z) = \frac{1}{1 - z^{-1}}$$

$$x_c[n] \rightarrow X_c(z) = \Delta(z) - U(z)$$

$$= 1 - \frac{1}{1 - z^{-1}} = \frac{-z^{-1}}{1 - z^{-1}}$$

10.4




a) $H_1(z)$:

$$y_1[n] = 4\,x_1[n] + 4\,x_1[n-2]$$

$$Y_1(z) = 4\,X_1(z) + z^{-2}\,4\,X_1(z) \Rightarrow H_1(z) = 4 + 4z^{-2}$$

$H_2(z)$:

$$Y_2[n] = -3\,X_2[n] - 3\,X_2[n-1]$$

$$Y_2[z] = -3\,X_2(z) - 3z^{-1}X[z] \Rightarrow H_2(z) = -3(1+z^{-1})$$

$H_3(z)$:

$$H_3(z) = 5(z^{-1} - z^{-2})$$

b)

$$\begin{aligned}
H(z) &= H_1(z)\,H_2(z)\,H_3(z) \\
&= 4(1+z^{-2})\left[-3(1+z^{-1})\right]\left[5(z^{-1}-z^{-2})\right] \\
&= -60(1+z^{-2})(1+z^{-1})(1-z^{-1})\,z^{-1} \\
&= -60z^{-1}(1+z^{-2})(1-z^{-2}) \\
&= -60z^{-1}(1-z^{-4}) = -60z^{-1} + 60z^{-5}
\end{aligned}$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = -60z^{-1} + 60z^{-5}$$

$$\Rightarrow Y(z) = -60z^{-1}X(z) + 60z^{-5}X(z)$$

$$Y[n] = -60\,x[n-1] + 60\,x[n-5]$$

## PROBLEM 10.4*:

***This is the solution for the version with a typo for*** $\mathcal{S}_3$

Suppose that three systems are hooked together in "cascade." In other words, the output of $\mathcal{S}_1$ is the input to $\mathcal{S}_2$, and the output of $\mathcal{S}_2$ is the input to $\mathcal{S}_3$. The three systems are specified as follows:

$$\begin{aligned}
\mathcal{S}_1: &\qquad y_1[n] = 4x_1[n] + 4x_1[n-2] \\
\mathcal{S}_2: &\qquad y_2[n] = -3x_2[n] - 3x_2[n-1] \\
\mathcal{S}_3: &\qquad y_3[n] = 5x_3[n-1] - 5x[n-2]
\end{aligned}$$

*Note:* the output of $\mathcal{S}_i$ is $y_i[n]$ and the input is $x_i[n]$.

Determine the equivalent system that is a single operation from the input $x[n]$ (into $\mathcal{S}_1$) to the output $y[n]$ which is the output of $\mathcal{S}_3$. Thus $x[n]$ is $x_1[n]$ and $y[n]$ is $y_3[n]$.

(a) Determine the $z$-transform system function $H_i(z)$ for each system.

$$\begin{aligned}
H_1(z) &= 4 + 4z^{-2} \\
H_2(z) &= -3 - 3z^{-1} \\
H_3(z) &= \text{not well defined because there are two inputs} \\
Y_3(z) &= 5z^{-1}X_3(z) - 5z^{-2}X(z) \\
&= 5z^{-1}H_1(z)H_2(z)X(z) - 5z^{-2}X(z) \\
&= (5z^{-1}H_1(z)H_2(z) - 5z^{-2})X(z)
\end{aligned}$$

(b) Write *one difference equation* that defines the overall system in terms of $x[n]$ and $y[n]$ only.

$$\begin{aligned}
Y(z) = Y_3(z) &= H(z)X(z) \\
&= (5z^{-1}H_1(z)H_2(z) - 5z^{-2})X(z)
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow H(z) &= 5z^{-1}H_1(z)H_2(z) - 5z^{-2} \\
H(z) &= 5z^{-1}(4 + 4z^{-2})(-3 - 3z^{-1}) - 5z^{-2} \\
&= -60(z^{-1} + z^{-2} + z^{-3} + z^{-4}) - 5z^{-2} \\
&= -60z^{-1} - 65z^{-2} - 60z^{-3} - 60z^{-4}
\end{aligned}$$

$$\Longrightarrow y[n] = -60x[n-1] - 65x[n-2] - 60x[n-3] - 60x[n-4]$$

10.5 : a) $H(z) = H_1(z)\, H_2(z)\, H_3(z)$

$$= (z^{-2}+z^{-3})(6-3z^{-1})\,\frac{2}{8-10z^{-1}+3z^{-2}}$$

$$= 3\,(z^{-2}+z^{-3})(2-z^{-1})\,\frac{2}{(-2+z^{-1})(-4+3z^{-1})}$$

$$= \frac{-6\,(z^{-2}+z^{-3})}{-4+3z^{-1}} = \frac{6\,(z^{-2}+z^{-3})}{4-3z^{-1}}$$

b) Note, if $\bar{H}(z) = \frac{b}{1-az^{-1}} \Rightarrow \bar{h}[n] = b\,a^n u[n]$

so we need to convert $\frac{6(z^{-2}+z^{-3})}{4-3z^{-1}}$ into a "friendlier" form

$$H(z) = \frac{6(z^{-2}+z^{-3})}{4-3z^{-1}} = \frac{6/4\,(z^{-2}+z^{-3})}{1-\frac{3}{4}z^{-1}}$$

$$Y(z) = \frac{6/4\,(z^{-2}+z^{-3})}{1-\frac{3}{4}z^{-1}}\,X(z)$$

$$Y[n] = \frac{3}{4}Y[n-1] + \frac{6}{4}\left[x[n-2]+x[n-3]\right]$$

Find $h[n]$, let $x[n] = \delta[n]$

| $n$ | $Y[n] = h[n]$ |
|---|---|
| 0 | $Y(0) = \frac{3}{4}Y(-1) + \frac{6}{4}[\,0\,] = 0$ |
| 1 | $Y(1) = 0$ |
| 2 | $Y(2) = \frac{3}{4}Y(1) + \frac{6}{4}\cdot[1+0] = 6/4$ |
| 3 | $Y(3) = \frac{3}{4}\cdot\frac{6}{4} + \frac{6}{4}[0+1] = \frac{18}{16}+\frac{6}{4} = G$ |
| 4 | $Y(4) = \frac{3}{4}\cdot k + 0 = \frac{3}{4}G$ |
| 5 | $Y(5) = \frac{3}{4}(\frac{3}{4}k) + 0 = (\frac{3}{4})^2 G$ |

$$\Rightarrow \boxed{h[n] = \frac{6}{4}\delta(n-2) + \left(\frac{3}{4}\right)^{n-3} G'\, u[n-3]}$$

$G = \left(\frac{18}{16}+\frac{6}{4}\right)\left(\frac{3}{4}\right)^{-3} = \frac{56}{9}$

$\alpha = 3/4$