PROBLEM SET #11 SOLUTIONS, ECE 2025 SPRING 2003

PROBLEM 11.1. To do this problem we will exploit the fact that the Z-transform of $a^n u[n]$ is $\frac{1}{1 - az^{-1}}$.

(a) The last form is most convenient, since both terms can be converted to the time domain by inspection:

$$H_a(z) = 4 - \frac{3}{1 - 0.5z^{-1}} \longleftrightarrow h_a[n] = 4\delta[n] - 3(0.5)^n u[n].$$

(b) Again, the last term is most convenient:

$$H_b(z) = \frac{1}{1 + 0.9e^{j\pi/3}z^{-1}} + \frac{1}{1 + 0.9e^{-j\pi/3}z^{-1}} \longleftrightarrow h_b[n] = (-0.9e^{j\pi/3})^n u[n] + (-0.9e^{-j\pi/3})^n u[n]$$

$$= (e^{jn\pi/3} + e^{-jn\pi/3})(-0.9)^n u[n]$$

$$= 2\cos\left(\frac{\pi}{3}n\right)(-0.9)^n u[n] .$$

Equivalently, since $(-0.9e^{j\pi/3}) = 0.9e^{-j2\pi/3}$, an alternative representation of the solution is:

$$h_b[n] = 2\cos\left(\frac{2\pi}{3}n\right)(0.9)^n u[n] .$$

(c) Again, the last term is most convenient:

$$H_c(z) = 32 - \frac{7}{1 - \frac{\sqrt{3}}{2}e^{j\pi/2}z^{-1}} - \frac{7}{1 - \frac{\sqrt{3}}{2}e^{-j\pi/2}z^{-1}} \longleftrightarrow h_c[n] = 32\delta[n] - 7(\tfrac{\sqrt{3}}{2}e^{j\pi/2})^n u[n] - 7(\tfrac{\sqrt{3}}{2}e^{-j\pi/2})^n u[n]$$

$$= (e^{jn\pi/2} + e^{-jn\pi/2})\left(\frac{\sqrt{3}}{2}\right)^n u[n]$$

$$= 2\cos\left(\frac{\pi}{2}n\right)\left(\frac{\sqrt{3}}{2}\right)^n u[n] .$$

PROBLEM 11.2. This problem considers difference equations of the following form:

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + b_3 x[n-3] + b_4 x[n-4].$$

We can find a general formula for the system function $H(z)$ by taking the Z-transform and solving for the ratio $H(z) = Y(z)/X(z)$. Specifically, moving all of the $y[\cdot]$ terms to the left-hand side and taking the Z-transform of both sides yields:

$$Y(z) - a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + b_3 z^{-3} X(z) + b_4 z^{-4} X(z)$$

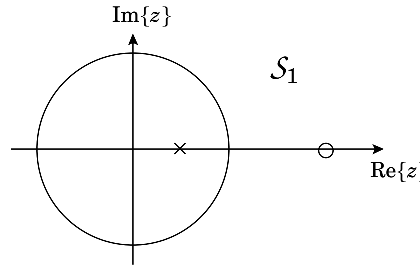
$$Y(z)\left(1 - a_1 z^{-1} - a_2 z^{-2}\right) = X(z)\left(b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4}}{1 - a_1 z^{-1} - a_2 z^{-2}} .$$

(a)

$$H_1(z) = \frac{1-2z^{-1}}{1-0.5z^{-1}}$$

$$= \frac{z-2}{z-0.5} \implies \text{zero at 2, pole at 0.5:}$$



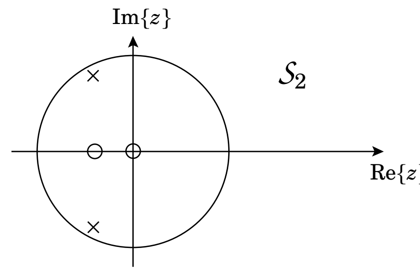
(b)

$$H_2(z) = \frac{2+0.9z^{-1}}{1+0.9z^{-1}+0.81z^{-2}}$$

$$= \frac{z(2z+0.9)}{z^2+0.9z+0.81}$$

$$= \frac{2z(z+0.45)}{(z+0.9e^{j\pi/3})(z+0.9e^{-j\pi/3})}$$

$$\implies \text{zeros at 0 and } -0.45,$$
$$\text{poles at } -0.9e^{\pm j\pi/3}:$$



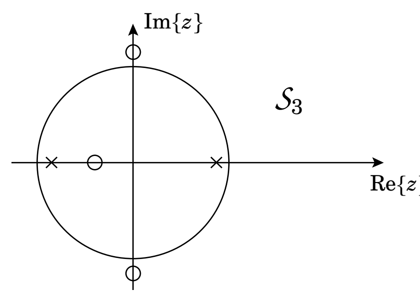
(c)

$$H_3(z) = \frac{18+24z^{-2}}{1-0.75z^{-2}}$$

$$= \frac{18z^2+24}{z^2-0.75} = \frac{18(z^2+\frac{4}{3})}{(z-\frac{\sqrt{3}}{2})(z-\sqrt{\frac{3}{2}})}$$

$$= \frac{18(z-\frac{2}{\sqrt{3}}e^{j\pi/2})(z-\frac{2}{\sqrt{3}}e^{-j\pi/2})}{(z-\frac{\sqrt{3}}{2})(z+\frac{\sqrt{3}}{2})}$$

$$\implies \text{zeros at } \pm\tfrac{2j}{\sqrt{3}}, \text{ poles at } \pm\tfrac{\sqrt{3}}{2}:$$



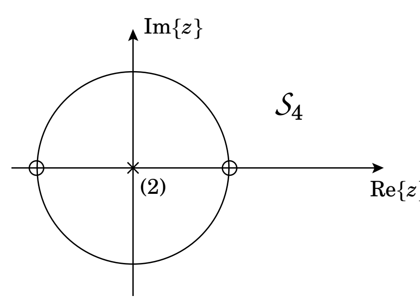
(d)

$$H_3(z) = \frac{1-z^{-4}}{1+z^{-2}} = = \frac{z^4-1}{z^4+z^2}$$

$$= \frac{z^4-1}{z^2(z^2+1)} = \frac{(z^2-1)\cancel{(z^2+1)}}{z^2\cancel{(z^2+1)}} \text{ (cancellation)}$$

$$= \frac{z^2-1}{z^2} = \frac{(z-1)(z+1)}{z^2}$$

$$\implies \text{zeros at } \pm 1, \text{ two poles at 0 :}$$



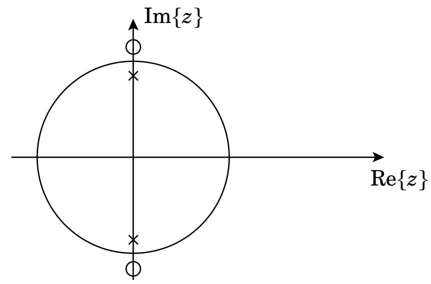
PROBLEM 11.3.

(a) $$H(z) = \frac{18 + 24z^{-2}}{1 + 0.75z^{-2}} = \frac{18z^2 + 24}{z^2 + 0.75}$$

$$= \frac{18(z^2 + \frac{4}{3})}{z^2 + 0.75} = \frac{18(z + \frac{2}{\sqrt{3}}e^{j\pi/2})(z + \frac{2}{\sqrt{3}}e^{-j\pi/2})}{(z + \frac{\sqrt{3}}{2}e^{j\pi/2})(z + \frac{\sqrt{3}}{2}e^{-j\pi/2})}$$

$\Longrightarrow$ zeros at $\frac{2}{\sqrt{3}}e^{\pm j\pi/2}$ (or equivalently, at $\pm\frac{2j}{\sqrt{3}}$),

and poles at $\frac{\sqrt{3}}{2}e^{\pm j\pi/2}$ (or equivalently, at $\pm\frac{\sqrt{3}j}{2}$):



(b) Before taking inverse Z-transform, rearrange $H(z)$ as follows:

$$H(z) = \frac{18 + 24z^{-2}}{1 + 0.75z^{-2}} = 18\left(\frac{1 + \frac{4}{3}z^{-2}}{1 + \frac{3}{4}z^{-2}}\right).$$

The partial fraction expansion procedure is simplest when the numerator degree is smaller than the denominator, so we will modify $H(z)$ before proceeding. Specifically, multiply the numerator by $9/16$, and divide the coefficient 18 by the same amount, so that $H(z)$ becomes:

$$H(z) = \frac{18}{9/16}\left(\frac{\frac{9}{16} + \frac{3}{4}z^{-2}}{1 + \frac{3}{4}z^{-2}}\right) = 32\left(\frac{\frac{9}{16} + \frac{3}{4}z^{-2}}{1 + \frac{3}{4}z^{-2}}\right).$$

Now we add and subtract $7/16$ to numerator, yielding:

$$H(z) = 32\left(\frac{\frac{9}{16} + \frac{7}{16} - \frac{7}{16} + \frac{3}{4}z^{-2}}{1 + \frac{3}{4}z^{-2}}\right) = 32\left(\frac{1 + \frac{3}{4}z^{-2}}{1 + \frac{3}{4}z^{-2}} - \frac{\frac{7}{16}}{1 + \frac{3}{4}z^{-2}}\right) = 32\left(1 - \frac{\frac{7}{16}}{1 + \frac{3}{4}z^{-2}}\right)$$

$$= 32 - \frac{14}{1 + \frac{3}{4}z^{-2}} = 32 - \frac{14}{(1 + \frac{\sqrt{3}}{2}jz^{-1})(1 - \frac{\sqrt{3}}{2}jz^{-1})}$$

$$= 32 - \left(\frac{A}{1 + \frac{\sqrt{3}}{2}jz^{-1}} + \frac{B}{1 - \frac{\sqrt{3}}{2}jz^{-1}}\right)$$

$$= 32 - \frac{(A + B) + (B - A)\frac{\sqrt{3}}{2}z^{-1}}{(1 + \frac{\sqrt{3}}{2}jz^{-1})(1 - \frac{\sqrt{3}}{2}jz^{-1})}.$$

To have equality, the numerator above must be 14. This leads to two equations and two unknowns:

$$A + B = 14\,,$$
$$B - A = 0\,.$$

The second equation implies $B = A$, which with the first equation implies $A = B = 7$. Thus:

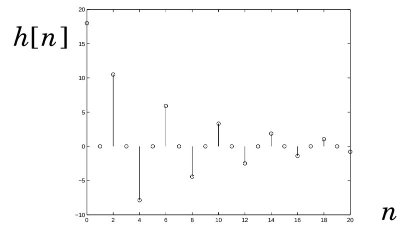
$$H(z) = 32 - 7\left(\frac{1}{1 + \frac{\sqrt{3}}{2}jz^{-1}} + \frac{1}{1 - \frac{\sqrt{3}}{2}jz^{-1}}\right)$$

Taking the inverse Z-transform yields:

$$h[n] = 32\delta[n] - 7\left(\left(\tfrac{\sqrt{3}}{2}e^{-j\pi/2}\right)^n u[n] + \left(\tfrac{\sqrt{3}}{2}e^{j\pi/2}\right)^n u[n]\right)$$

$$= 32\delta[n] - 7\left(\tfrac{\sqrt{3}}{2}\right)^n \left(e^{-jn\pi/2} + e^{jn\pi/2}\right) u[n]$$

$$= 32\delta[n] - 14\left(\tfrac{\sqrt{3}}{2}\right)^n \cos(n\pi/2) u[n] \Longrightarrow$$



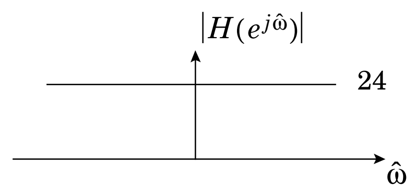
(c) We are told that the system is causal, and we observe from part (a) that the poles are inside the unit circle. Therefore, the system is stable.

(d) To get the frequency response, evaluate $H(z)$ on the unit circle:

$$H(z) = \frac{18 + 24z^{-2}}{1 + 0.75z^{-2}} \Longrightarrow H(e^{j\hat{\omega}}) = \frac{18 + 24e^{-2j\hat{\omega}}}{1 + 0.75e^{-2j\hat{\omega}}} = 24e^{-2j\hat{\omega}}\left(\frac{1 + 0.75e^{2j\hat{\omega}}}{1 + 0.75e^{-2j\hat{\omega}}}\right).$$

(e) Taking the absolute value of the last expression above leads to the magnitude response:

$$\left|H(e^{j\hat{\omega}})\right| = 24\left|e^{-2j\hat{\omega}}\right| \left|\frac{1 + 0.75e^{2j\hat{\omega}}}{1 + 0.75e^{-2j\hat{\omega}}}\right| = 24 \Longrightarrow$$



(The magnitude of the numerator and denominator are the same because one is the complex-conjugate of the other, and the conjugate does not change the absolute value.)
A filter whose magnitude response is a constant for all frequency is called an *all-pass* filter.

(f) The response of an LTI system to an input of $2\cos(\hat{\omega}_1 n)$ is $2A\cos(\hat{\omega}_1 n + \phi)$, where A and ϕ are the magnitude and phase of the frequency response $H(e^{j\hat{\omega}})$ evaluated at $\hat{\omega}_1$, *i.e.,*

$$Ae^{j\phi} = H(e^{j\hat{\omega}_1}) .$$

In this case we have $\hat{\omega}_1 = \pi/2$, so that:

$$Ae^{j\phi} = H(z)\big|_{z = e^{j\pi/2}} = \frac{18 + 24z^{-2}}{1 + 0.75z^{-2}}\bigg|_{z^2 = -1} = \frac{18 - 24}{1 - 0.75} = -24.$$

The magnitude is 24, as expected, while the phase is $\phi = \pi$.
Therefore, the final answer is $y_1[n] = 2A\cos(\hat{\omega}_1 n + \phi) = -48\cos(0.5\,\pi n)$.

PROBLEM 11.4.

(J) This is FIR, so it is one of system $\mathcal{S}_4$ through $\mathcal{S}_8$. Of these, only $\mathcal{S}_4$ will produce an impulse response with 4 nonzero terms.

(K) This is IIR, so it is one of system $\mathcal{S}_0$ through $\mathcal{S}_3$, or $\mathcal{S}_9$. Of these, only $\mathcal{S}_3$ will produce a value of $h[0] = 0.5$ at time zero.

(L) This is IIR, so it is one of system $\mathcal{S}_0$ through $\mathcal{S}_3$, or $\mathcal{S}_9$. Furthermore, the impulse starts at time $n = 2$, not time zero, with a value of $h[2] = 1.5$. That leads to system $\mathcal{S}_0$.

(M) This is FIR with coefficients that ramp up and then down, just as prescribed by $\mathcal{S}_8$.

(N) This is IIR that starts at time 1 with a value of $h[1] = 5$, which leads us to system $\mathcal{S}_1$.

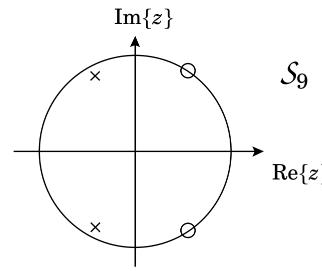
(O) This is FIR with coefficients alternating between 3 and –3, as prescribed by $\mathcal{S}_5$.

PROBLEM 11.5.

(A) $H(e^{j\hat{\omega}})$ has 7 zeros on the unit circle, and so the number of $H(z)$ must be a polynomial in z^{-1} of order 7. Only $\mathcal{S}_7$ fits this description.

(B) This is a rough highpass filter with $H(e^{j0}) = H(1) \approx 3$. This matches $\mathcal{S}_1$, since in that case $H_1(z) = \frac{5z^{-1}}{1 + 0.7z^{-1}}$, which is highpass, and which satisfies $H_1(1) = \frac{5}{1 + 0.7} = 2.941$.

(C) $H(e^{j\hat{\omega}})$ is zero near $\hat{\omega} = \pm 0.4\pi$, and $H(e^{j\hat{\omega}})$ is large near $\hat{\omega} = \pm 0.6\pi$. This matches $\mathcal{S}_9$, which has the following pole-zero plot:

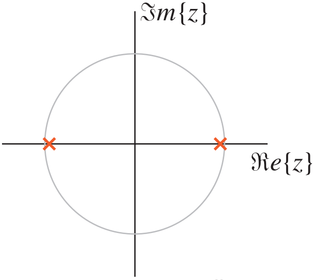


(D) Frequency response D and E are roughly similar, in that both are roughly high-pass filters and both have a zero at d.c. However, D has a narrow bandpass region that is characteristic of an IIR filter (specifically, with a pole close to the unit circle), whereas E has a broad bandpass region that is characteristic of an FIR filter. Both $\mathcal{S}_3$ and $\mathcal{S}_4$ are highpass filters with a zero at d.c., but because $\mathcal{S}_3$ is IIR, it matches D.

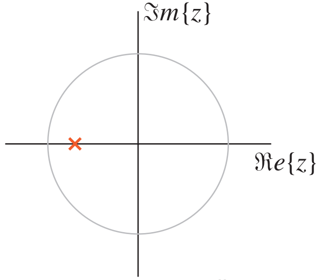
(E) Because of the above reasoning, $\mathcal{S}_4$ matches E.

(F) The magnitude response is roughly 15 at both $z = 1$ and $z = -1$ (i.e., both $\hat{\omega} = 0$ and $\hat{\omega} = \pi$), and this is matched by $\mathcal{S}_0$, which has system function:

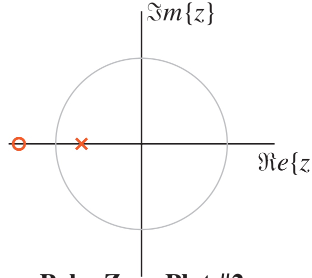
$H_0(z) = \frac{1.5z^{-2}}{1 - 0.9z^{-2}}$, and hence satisfies $H_1(1) = \frac{1.5}{1 - 0.9} = 15$, and also $H_1(-1) = \frac{1.5}{1 - 0.9} = 15$.



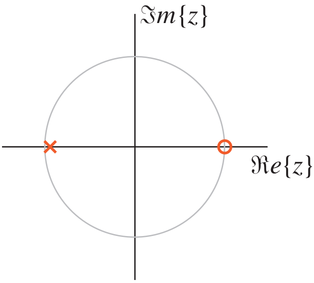
Pole–Zero Plot #0



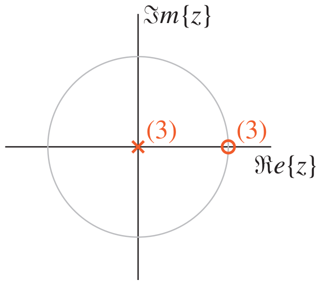
Pole–Zero Plot #1



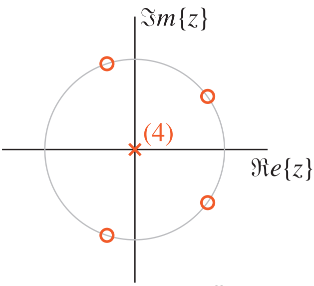
Pole–Zero Plot #2



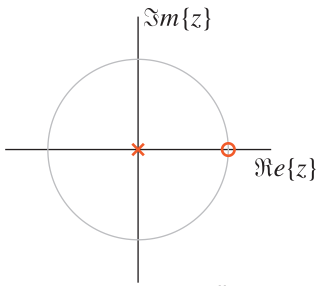
Pole–Zero Plot #3



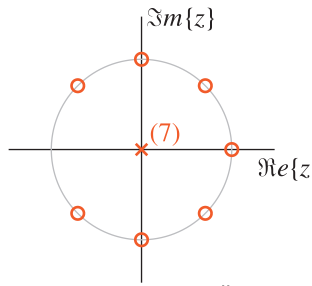
Pole–Zero Plot #4



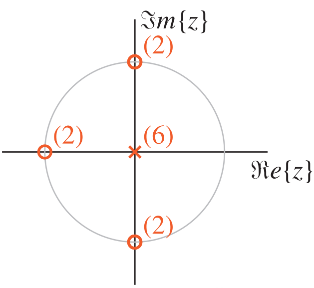
Pole–Zero Plot #5



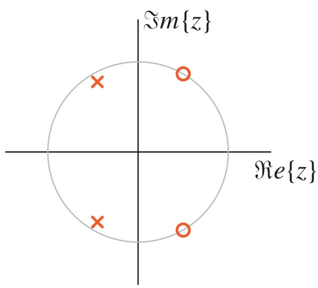
Pole–Zero Plot #6



Pole–Zero Plot #7



Pole–Zero Plot #8



Pole–Zero Plot #9

PROBLEM 11.6. From the difference equation, the system function is:

$$H(z) = \frac{1+z^{-2}}{1-0.8z^{-1}} = \frac{1}{1-0.8z^{-1}} + \frac{z^{-2}}{1-0.8z^{-1}} = H_1(z) + H_2(z).$$

(a) We will use approach #2 to find the step response for each of the above terms separately, and then add the results to get the overall step response. The Z-transform of the unit step is $\frac{1}{1-z^{-1}}$, so the Z-transform of the step response to the first system is:

$$Y_1(z) = \frac{1}{(1-0.8z^{-1})(1-z^{-1})}$$

$$= \frac{A}{1-0.8z^{-1}} + \frac{B}{1-z^{-1}}$$

$$= \frac{(A+B)-(A+0.8B)z^{-1}}{(1-0.8z^{-1})(1-z^{-1})} \quad \Longrightarrow A = -4,\ B = 5.$$

$$= \frac{-4}{1-0.8z^{-1}} + \frac{5}{1-z^{-1}} \quad \longleftrightarrow \quad y_1[n] = (5 - 4(0.8)^n)u[n].$$

The step response of the second system $H_2(z)$ will be the same, but delayed by two. Therefore, the overall step response is:

$$y[n] = y_1[n] + y_1[n-2] = (5 - 4(0.8)^n)u[n] + (5 - 4(0.8)^{n-2})u[n-2].$$

(b) Use approach #3. The response of an LTI system to an input of $\cos(\hat{\omega}_1 n + \theta)$ is $A\cos(\hat{\omega}_1 n + \theta + \phi)$, where A and ϕ satisfy $Ae^{j\phi} = H(e^{j\hat{\omega}_1})$. But:

$$H(e^{j0.5\pi}) = \left.\frac{1+z^{-2}}{1-0.8z^{-1}}\right|_{z=j} = \frac{1-1}{1+0.8j} = 0.$$

$$H(e^{j0.25\pi}) = \left.\frac{1+z^{-2}}{1-0.8z^{-1}}\right|_{z=e^{j0.25\pi}} = \frac{1-j}{1-0.8e^{-j0.25\pi}} \approx 1.983e^{-j0.542\pi}.$$

Hence, the response to

$$x[n] = 2\cos(0.5\pi n - \pi/2) + \cos(0.25\pi n - \pi)$$

is

$$y[n] = 0 + 1.983\cos(0.25\pi n - \pi - 0.542\pi)$$
$$= 1.983\cos(0.25\pi n + 0.458\pi).$$

(c) In terms of the impulse response $h[n]$, the response to $10\delta[n-5]$ will be $10h[n-5]$. From part (a), the impulse response is:

$$\frac{1}{1-0.8z^{-1}} + \frac{z^{-2}}{1-0.8z^{-1}} \longleftrightarrow h[n] = (0.8)^n u[n] + (0.8)^{n-2}u[n-2]$$

$$= \delta[n] + 0.8\delta[n-1] + (1+(0.8)^{-2})(0.8)^n u[n-2]$$

$$= \delta[n] + 0.8\delta[n-1] + 2.5625(0.8)^n u[n-2].$$

Therefore, $y[n] = 10h[n-5] = 10\delta[n-5] + 8\delta[n-6] + 25.625(0.8)^{n-5}u[n-7]$.

(d) Use approach #1.