Solutions To Problem Set #12 (MH)

Problem 12.1

a) $\left[2e^{-200t}u(t) + \sin(100\pi t)\,u(t)\right]\cdot \delta(t-0.005)$

$= \left[2e^{-200(0.005)} + \sin(100\pi\cdot 0.005)\right]\cdot \delta(t-0.005) = (2e^{-1}+1)\cdot\delta(t-0.005)$

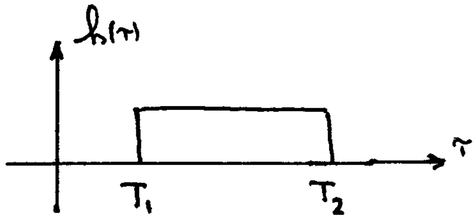
b) $\left[\delta(t-0.2) + \delta(t+0.2)\right] * \delta(t-5) = \delta(t-5.2) + \delta(t-4.8)$

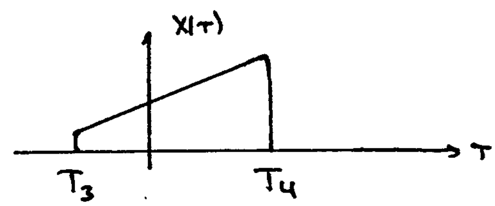
c) $\frac{d}{dt}\left\{e^{5t}\left[u(t+3) - u(t-3)\right]\right\} = 5e^{5t}\left[u(t+3)-u(t-3)\right] + e^{-15}\delta(t+3) - e^{15}\delta(t-3)$

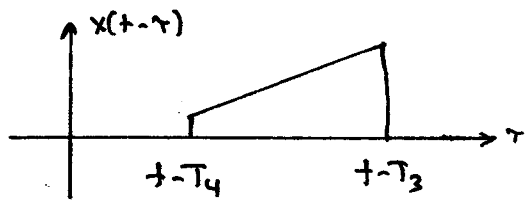
d) $\int_{-\infty}^{t-0.001} \delta(\tau-0.002)\sin(100\pi\tau)\,u(\tau)\,d\tau$

$= \int_{-\infty}^{t-0.001} \delta(\tau-0.002)\sin(0.2\pi)\,d\tau = \begin{cases} \sin(0.2\pi) & \text{if } t > 0.003 \\ 0 & \text{if } t \le 0.003 \end{cases}$

Problem 12.2







If we look at the product, $h(\tau)\cdot x(t-\tau)$, we note that

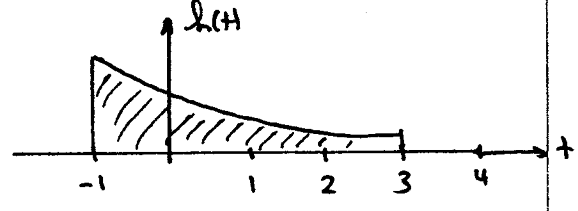
1) If $t-T_3 < T_1 \Rightarrow t < T_1+T_3$ then there is no overlap between $x(t-\tau)$ and $h(\tau)$, so the product is zero and the convolution is zero.

2) If $t-T_4 > T_2 \Rightarrow t > T_2+T_4$, the same holds true (no overlap, product is zero, and convolution is zero).

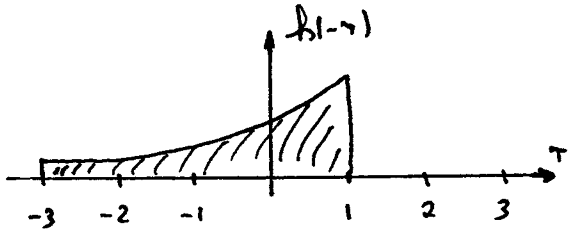
Conclusion: Convolution is zero except (possibly) for $T_1+T_3 < t < T_2+T_4$

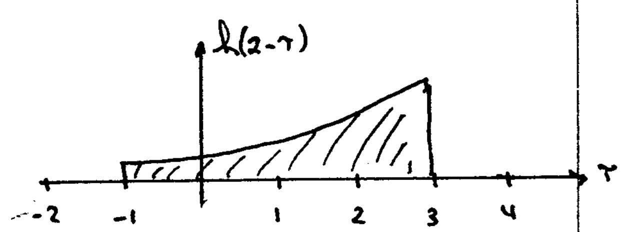
Problem 12.3

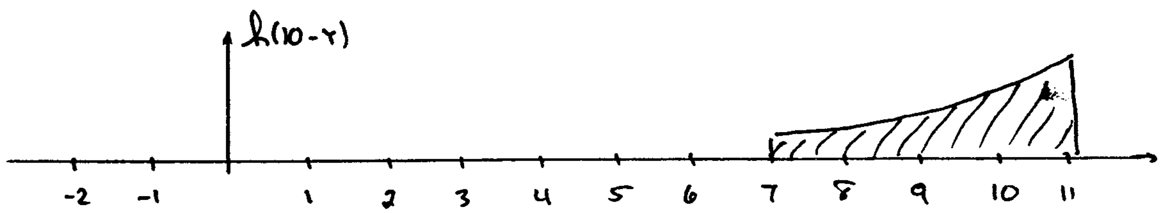
$$h(t) = \begin{cases} e^{-0.55t} & : -1 \le t \le 3 \\ 0 & : \text{else} \end{cases}$$



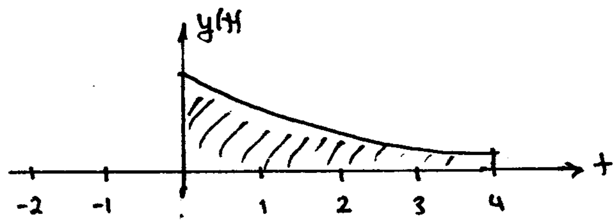
a)





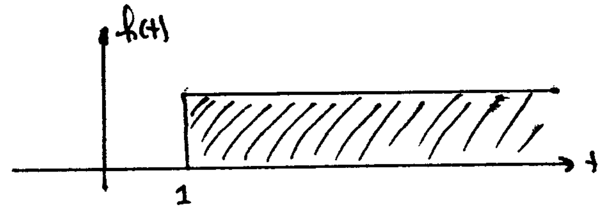


b) If $x(t) = \delta(t-1)$ then $y(t) = h(t-1)$

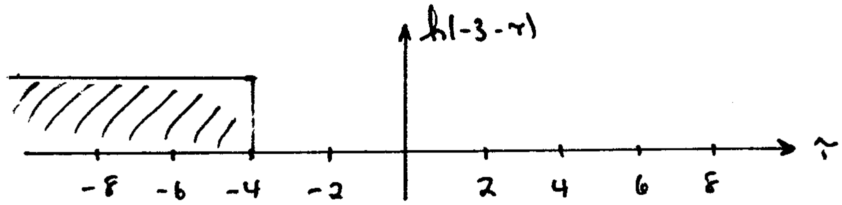


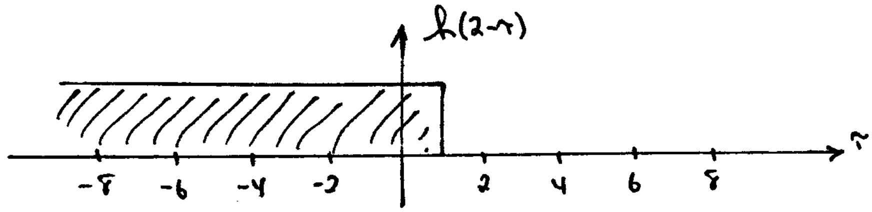
c) (see next Page)

Problem 12.4 : $h(t) = u(t-1)$



a)





Problem 12.3 Part (c)

A linear time-invariant system has impulse response:

$$h(t) = e^{-0.55t}\{u(t+1) - u(t-3)\} = \begin{cases} e^{-0.55t} & -1 \le t < 3 \\ 0 & \text{otherwise} \end{cases}$$

Use the convolution integral to determine the output $y(t)$ when the input is

$$x(t) = e^{-0.25t}\{u(t) - u(t-7)\} = \begin{cases} e^{-0.25t} & 0 \le t < 7 \\ 0 & \text{otherwise} \end{cases}$$

If you use the cconvdemo GUI, you will discover that the boundaries of the 5 regions are at $t = -1$, $t = 3$, $t = 6$ and $t = 10$. In each case, the integrand can be written as $h(\tau)x(t-\tau)$ if we flip $x(t)$. If we slide $x(t-\tau)$ along by changing t, we can view the overlap with $h(\tau)$ and obtain the the limits of integration as a function of t. Thus, the integrals become

$$y(t) = \begin{cases} 0 & t < -1 & \mathit{Region}\ 1 \\ \displaystyle\int_{-1}^{t} e^{-0.55\tau} e^{-0.25(t-\tau)} d\tau & -1 \le t < 3 & \mathit{Region}\ 2 \\ \displaystyle\int_{-1}^{3} e^{-0.55\tau} e^{-0.25(t-\tau)} d\tau & 3 \le t < 6 & \mathit{Region}\ 3 \\ \displaystyle\int_{t-7}^{3} e^{-0.55\tau} e^{-0.25(t-\tau)} d\tau & 6 \le t < 10 & \mathit{Region}\ 4 \\ 0 & 10 \le t & \mathit{Region}\ 5 \end{cases}$$

The general integrand that we must process is

$$\int_a^b e^{-0.55\tau} e^{-0.25(t-\tau)} d\tau = e^{-0.25t} \int_a^b e^{-0.3\tau} d\tau = e^{-0.25t}\left(\frac{e^{-0.3b} - e^{-0.3a}}{-3/10}\right)$$

Thus we obtain

$$y(t) = \begin{cases} 0 & t < -1 & \mathit{Region}\ 1 \\ \frac{10}{3}\left(e^{0.3}e^{-0.25t} - e^{-0.55t}\right) & -1 \le t < 3 & \mathit{Region}\ 2 \\ \frac{10}{3} e^{-0.25t}\left(e^{0.3} - e^{-0.9}\right) & 3 \le t < 6 & \mathit{Region}\ 3 \\ \frac{10}{3}\left(e^{2.1}e^{-0.55t} - e^{-0.9}e^{-0.25t}\right) & 6 \le t < 10 & \mathit{Region}\ 4 \\ 0 & 10 \le t & \mathit{Region}\ 5 \end{cases}$$

b) Yes, the system is causal since $h(t) = 0$ for $t < 0$.

c) No, the system is not stable since

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \infty$$

A (bounded) input that produces an unbounded output is $x(t) = u(t)$. In this case

$$y(t) = x(t) * h(t) = (t-1)u(t-1)$$

d) If $x(t) = u(t+2)$, then $x(t)$ "begins" at time $t = -2$. Since $h(t)$ begins at time $t = 1$, then $y(t) = x(t) * h(t)$ will be equal to zero for

$$t < t_1 = -2 + 1 = -1$$

e) To find the output signal for $t > t_1 = -1$ using the convolution integral, we have

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} u(\tau+2)\, u(t-\tau-1)\, d\tau$$

Note that the integrand is equal to one $\tau > -2$ and $\tau < t-1$, and it is equal to zero otherwise. Therefore, for $t > -1$ we have

$$y(t) = \int_{-2}^{t-1} d\tau = (t-1) - (-2) = t+1$$

$$\Rightarrow \quad \boxed{y(t) = (t+1)\, u(t+1)}$$

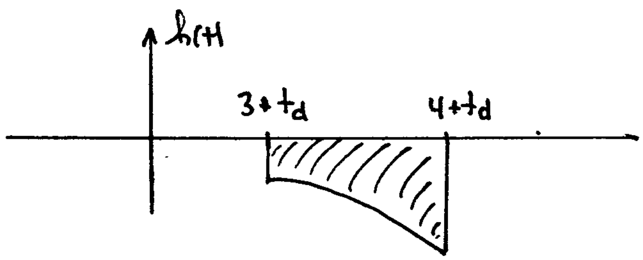
Problem 12.5

a) $h(t) = [h_1(t) - h_2(t)] * h_3(t)$

$$= [2\delta(t-4) - \delta(t-3)] * 2^{t} u(t-t_d)$$

$$= (2)2^{t-4} u(t-4-t_d) - 2^{t-3} u(t-3-t_d)$$

$$= 2^{t-3} [u(t-4-t_d) - u(t-3-t_d)]$$



b) If this system is to be causal, we require $h(t) = 0$ for $t < 0$. This will be true when $3 + t_d > 0$, or $t_d > -3$.

c) Systems #1 and #2 are clearly stable since

$$\int_{-\infty}^{\infty} |h_1(t)|\,dt = 2 \quad \text{and} \quad \int_{-\infty}^{\infty} |h_2(t)|\,dt = 1$$

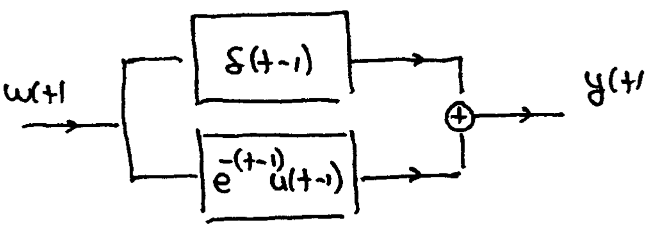
System #3, however, is unstable since $h_3(t) \xrightarrow[t \to \infty]{} \infty$

Nevertheless, the overall system is stable because $h(t)$ is non-zero only over a finite range of values for t.

Problem 12.6

a) $$y(t) = \underbrace{w(t-1)}_{\text{delay}} + \int_{-\infty}^{\infty} \underbrace{e^{-\tau+1}u(\tau-1)}_{h_3(\tau)}\, w(t-\tau)\, d\tau$$

we may view this system as a parallel connection of two LTI systems, an ideal delay and one with an impulse response $h_3(t) = e^{-(t-1)}u(t-1)$



so the impulse response we are looking for is: $h_3(t) = \delta(t-1) + e^{-(t-1)}u(t-1)$

b) $$h(t) = h_1(t) * h_2(t) = \left[u(t) - e^{-2t}u(t)\right] * \left[\delta(t-1) + e^{-(t-1)}u(t-1)\right]$$

Note that we may express the second term as follows:

$$\delta(t-1) + e^{-(t-1)}u(t-1) = \left[\delta(t) + e^{-t}u(t)\right] * \delta(t-1)$$

Therefore, lets ignore the convolution with $\delta(t-1)$ for now, and look at the other terms:

$$\left[u(t) - e^{-2t}u(t)\right] * \left[\delta(t) + e^{-t}u(t)\right]$$

$$= \underbrace{u(t) * \delta(t)}_{u(t)} + u(t) * e^{-t}u(t) - \underbrace{e^{-2t}u(t) * \delta(t)}_{-e^{-2t}u(t)} - e^{-2t}u(t) * e^{-t}u(t)$$

To evaluate the other two convolutions, we may use the following result from class:

$$e^{-at}u(t) * e^{-bt}u(t) = \frac{1}{b-a}\left[e^{-at} - e^{-bt}\right]u(t)$$

provided $a \neq b$. Using this, we have:

$$\underset{(a=0)}{u(t)} * \underset{(b=1)}{e^{-t}u(t)} = \left[1 - e^{-t}\right]u(t)$$

$$\underset{(a=2)}{e^{-2t}u(t)} * \underset{(b=1)}{e^{-t}u(t)} = -\left[e^{-2t} - e^{-t}\right]u(t)$$

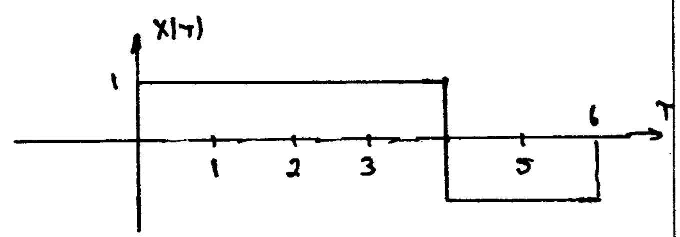
Putting this all together, we have (including the convolution with $\delta(t-1)$):

$$\left[u(t) - e^{-2t}u(t)\right] * \left[\delta(t) + e^{-t}u(t)\right] * \delta(t-1)$$

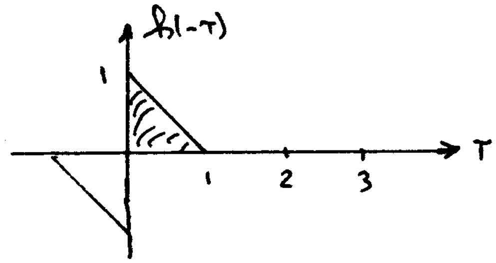
$$= u(t-1) - e^{-2(t-1)}u(t-1) + \left[1 - e^{-(t-1)}\right]u(t-1) + \left[e^{-2(t-1)} - e^{-(t-1)}\right]u(t-1)$$

Problem 12.7

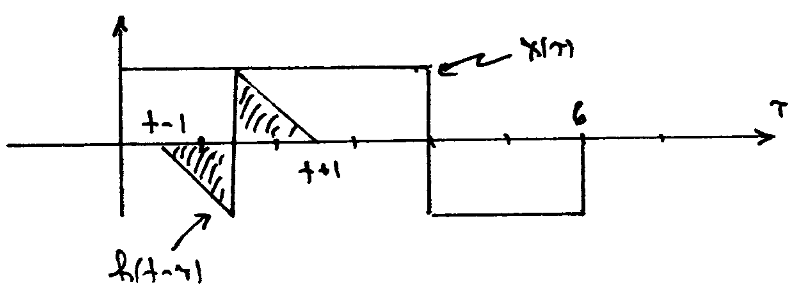
a) $y(0) = \int_{-\infty}^{\infty} x(\tau)h(-\tau)\,d\tau$



Note that $x(\tau)h(-\tau)$ is equal to zero for all τ except from $\tau=0$ to $\tau=1$ where it decreases linearly from a value of 1 to zero. Thus $y(0)$ is equal to the area of the signal shown shaded above, i.e., $y(0) = 1/2$.



b)



$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = 0$ when:

a) $t+1 < 0 \Rightarrow t < -1$

b) $t-1 > 6 \Rightarrow t > 7$

c) $t-1 > 0$ and $t+1 < 4 \Rightarrow 1 < t < 3$

d) $t-1 = 4 \Rightarrow t = 5$