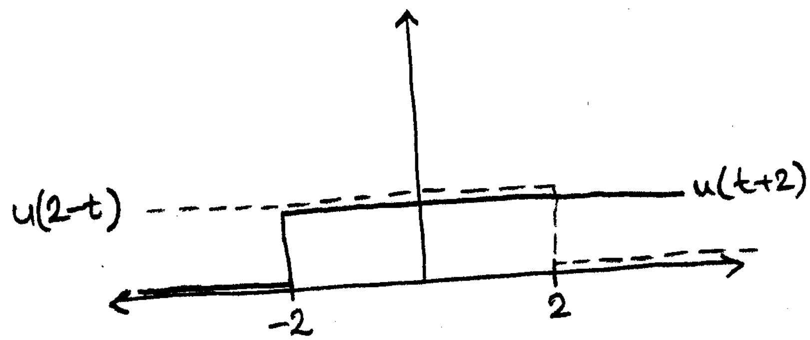
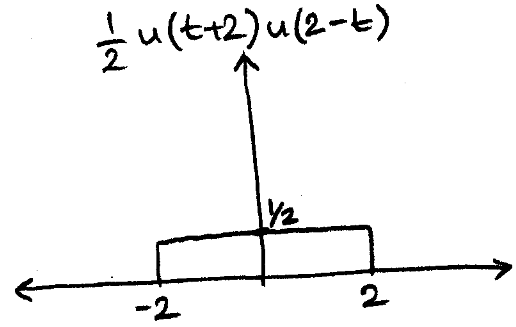
Solutions for Problem Set #14

14.1

(a) $x(t) = \frac{1}{2} u(t+2)\, u(2-t)$





$$x(t) = \frac{1}{2} u(t+2) u(2-t) = \frac{1}{2}\left[u(t+2) - u(t-2)\right]$$

$$X(j\omega) = \frac{\sin(2\omega)}{\omega}$$

(b) $x(t) = \frac{d}{dt}\left(\frac{\sin(20\pi t)}{10\pi t}\right)$

Let $y(t) = \frac{1}{10} \cdot \frac{\sin(20\pi t)}{\pi t}$. Then, $Y(j\omega) = \frac{1}{10}\left[u(\omega+20\pi) - u(\omega-20\pi)\right]$

$$X(j\omega) = (j\omega)\cdot Y(j\omega) = (j\omega)\cdot \frac{1}{10}\cdot\left[u(\omega+20\pi) - u(\omega-20\pi)\right]$$

(c) $x(t) = \frac{\sin(20\pi t)}{10\pi t} \cdot \cos(100\pi t)$

$$X(j\omega) = \frac{1}{2} Y(j(\omega-100\pi)) + \frac{1}{2} Y(j(\omega+100\pi))$$

Use $Y(j\omega)$ from part (b).

$$X(j\omega) = \frac{1}{20}\left[u(\omega-80\pi) - u(\omega-120\pi)\right] + \frac{1}{20}\left[u(\omega+120\pi) - u(\omega+80\pi)\right]$$

(d) $X(j\omega) = \dfrac{e^{-j2\omega}}{3+j4\omega}$

$X(j\omega) = e^{-j2\omega} \cdot \dfrac{1}{4} \cdot \dfrac{1}{\frac{3}{4}+j\omega}$

Let $V(j\omega) = \dfrac{1}{4} \cdot \dfrac{1}{\frac{3}{4}+j\omega}$. Then, $v(t) = \frac{1}{4} \cdot e^{-\frac{3}{4}t} \cdot u(t)$

$x(t) = v(t-2)$

$x(t) = \dfrac{1}{4} \cdot e^{-\frac{3}{4}(t-2)} \cdot u(t-2)$

(e) $X(j\omega) = \dfrac{j2\omega}{3+4j\omega}$

$X(j\omega) = \dfrac{2}{4} \cdot \dfrac{j\omega}{\frac{3}{4}+j\omega} = \dfrac{1}{2} \cdot j\omega \cdot \dfrac{1}{\frac{3}{4}+j\omega}$

$x(t) = \dfrac{1}{2} \dfrac{d}{dt} e^{-\frac{3}{4}t} u(t)$

$= \dfrac{1}{2} e^{-\frac{3}{4}t} \left(\delta(t) - \frac{3}{4} u(t) \right)$

(f) Let $\omega_0 = 20\pi$ and $a_k = e^{j\pi k}$. Then,

$X(j\omega) = \pi \sum_{k=-\infty}^{\infty} a_k \cdot \delta(\omega - k \cdot \omega_0)$

$x(t) = \dfrac{1}{2} \cdot \sum_{k=-\infty}^{\infty} a_k \cdot e^{jk\omega_0 t}$

$= \dfrac{1}{2} \sum_{k=-\infty}^{\infty} e^{j\pi k} \cdot e^{jk20\pi t}$

PROBLEM 14.1*:

(f) $X(j\omega) = \sum_{k=-\infty}^{\infty} \pi\, e^{j\pi k} \delta(\omega - 20\pi k)$ ***(Make a plot of $x(t)$ obtained in this part.)***

Solution: The regularly spaced impulses in ω represent the Fourier Series coefficients of a periodic signal in the time domain. Thus

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \pi\, e^{j\pi k} \delta(\omega - 20\pi k) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - 20\pi k)$$

Can you find the signal $x(t)$ that has the Fourier Series coefficients given by $a_k = \frac{1}{2}e^{j\pi k}$?

$$x(t) = \sum_{k=-\infty}^{\infty} a_k\, e^{j20\pi kt} = \sum_{k=-\infty}^{\infty} \tfrac{1}{2}e^{j\pi k}\, e^{j20\pi kt} = \tfrac{1}{2} \sum_{k=-\infty}^{\infty} e^{j20\pi k(t+1/20)} \tag{1}$$

The answer comes from the identity: $\sum_{n=-\infty}^{\infty} \delta(t - nT) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{j2\pi kt/T}$

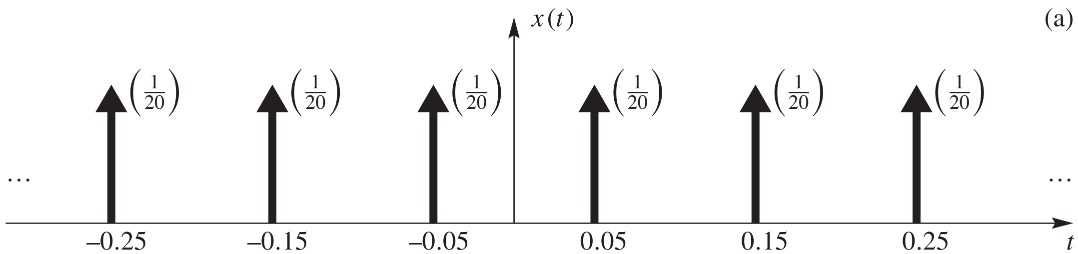
The value of T is $T = 1/10$, and we need the identity with t replaced by $(t + 1/20)$ in order to match Eq. (1).

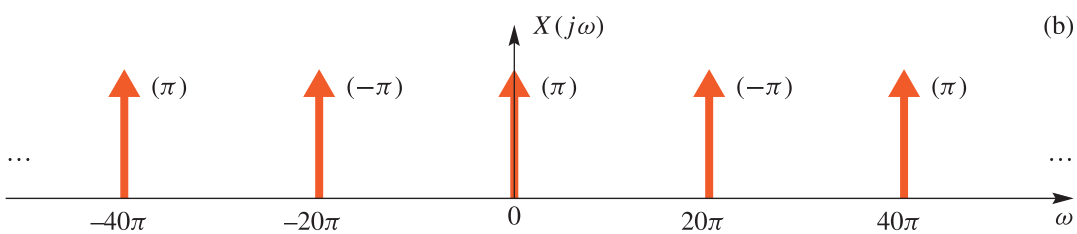
$$\sum_{n=-\infty}^{\infty} \delta(t + 1/20 - n/10) = 10 \sum_{k=-\infty}^{\infty} e^{j20\pi k(t+1/20)}$$

Finally, the signal $x(t)$ in (1) can be written as an impulse train:

$$x(t) = \sum_{n=-\infty}^{\infty} \tfrac{1}{20}\, \delta(t + 0.05 - 0.1n)$$

where the impulses are located at $t = \ldots, -0.15, -0.05, 0.05, 0.15, 0.25, \ldots$
From this equation, the plot of $x(t)$ is a shifted impulse train.





Note: Another way to get this answer is to use the ***Shifting Property***

$$x(t - t_d) \quad \longleftrightarrow \quad e^{-j\omega t_d}\, X(j\omega)$$

and apply it to $X(j\omega)$ rewritten as:

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \pi\, e^{j\pi k} \delta(\omega - 20\pi k) = e^{j\omega/20} \left(\sum_{k=-\infty}^{\infty} \pi\, \delta(\omega - 20\pi k) \right)$$

because $e^{j\omega/20}$ evaluated at $\omega = 20\pi k$ is $e^{j\pi k}$, from which we conclude that the time shift is $t_d = 1/20$ secs.

14.2

(a) From Figure (b) $\Rightarrow f_{max} = \frac{80\pi}{2\pi} = 40\ Hz$

From Sampling Theorem $\Rightarrow f_s \geq 2 \cdot f_{max} = 80$ samples/sec

$T_s = \frac{1}{80}$ sec $\qquad \omega_s = \frac{2\pi}{T_s} = 160\pi$

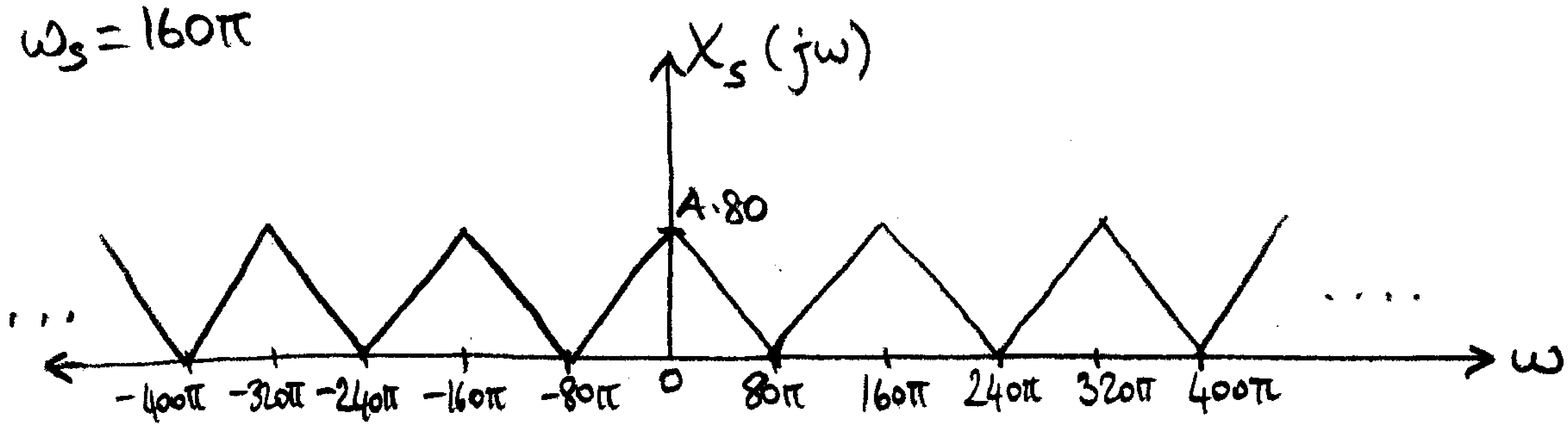
$$x_s(t) = x(t) \cdot p(t)$$

$$X_s(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

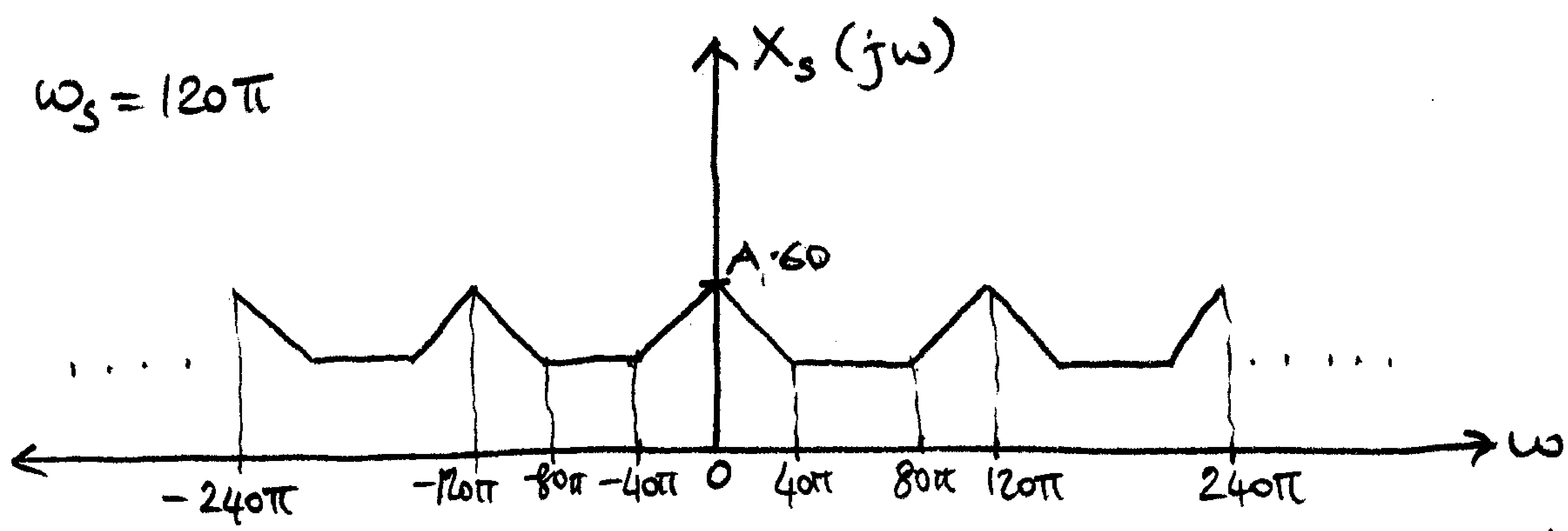
$$= \frac{1}{2\pi} X(j\omega) * \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi}{T_s} k\right)$$

$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j\omega - 160\pi k)$$

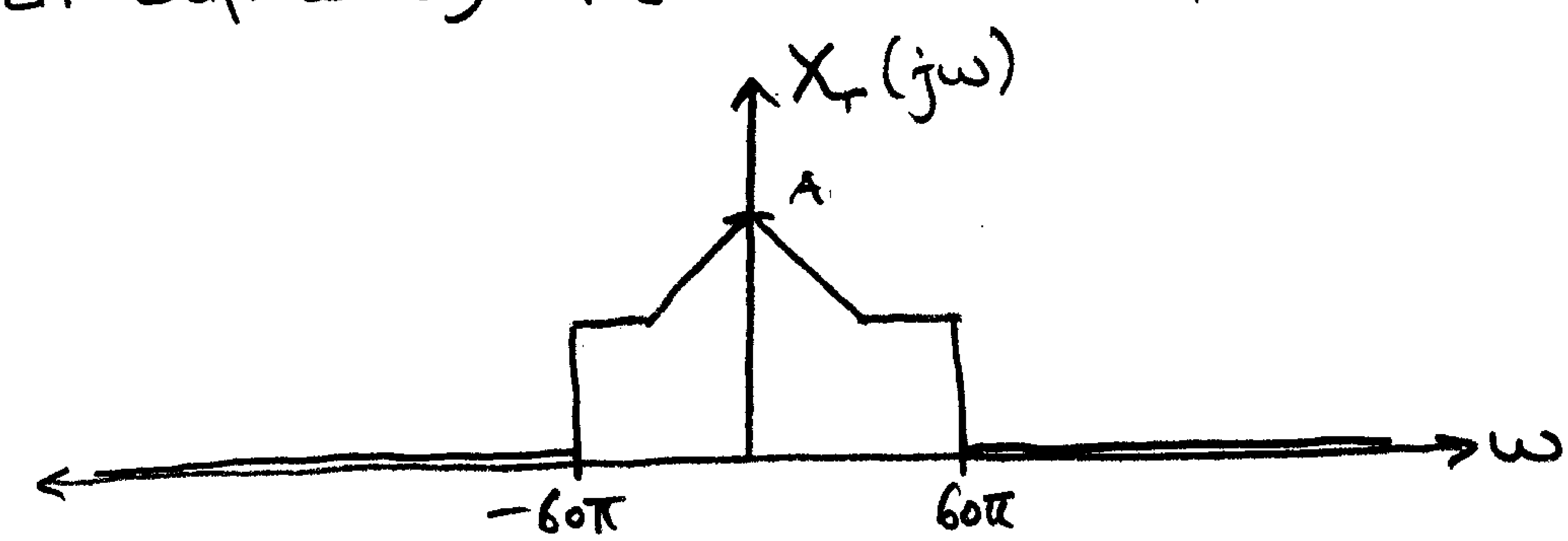
$\omega_s = 160\pi$



(b) $\omega_s = 120\pi$



(c) The LPF defined by $H_r(j\omega)$ removes all frequencies beyond 60π.



14.3

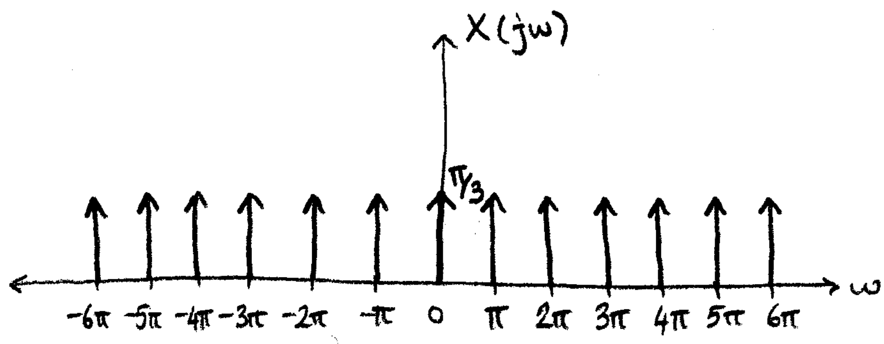
(a) $x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{3}\delta(t-2n)$

Use $\sum_{n=-\infty}^{\infty} \delta(t-nT) \xleftrightarrow{F} \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T}n)$

where $T=2$.

$$X(j\omega) = \frac{1}{3} \cdot \frac{2\pi}{2} \sum_{n=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{2}n)$$

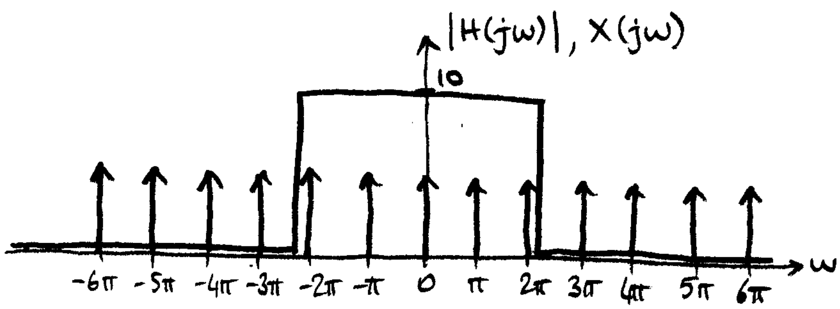
$$= \frac{\pi}{3} \sum_{n=-\infty}^{\infty} \delta(\omega - \pi n)$$



(b)

$$H(j\omega) = e^{-j\omega 0.1} \cdot 10[u(\omega+2.2\pi) - u(\omega-2.2\pi)] = \begin{cases} 10 \cdot e^{-j\omega 0.1}, & -2.2\pi \le \omega < 2.2\pi \\ 0, & o/w \end{cases}$$

$e^{-j\omega 0.1}$ does not contribute to $|H(j\omega)|$



(c)

$$Y(jw) = X(jw) \cdot H(jw) = \frac{10\pi}{3} \cdot e^{-jw0.1}\left(\delta(w+2\pi)+\delta(w+\pi)+\delta(w)+\delta(w-\pi)+\delta(w-2\pi)\right)$$

$$= \frac{5}{3} e^{-jw0.1}\left(2\pi\delta(w+2\pi)+2\pi\delta(w+\pi)+2\pi\delta(w-0)+2\pi\delta(w-\pi)+2\pi\delta(w-2\pi)\right)$$

$$y(t) = \frac{5}{3}\left(e^{-j2\pi(t-0.1)} + e^{-j\pi(t-0.1)} + e^{j0} + e^{j\pi(t-0.1)} + e^{j2\pi(t-0.1)}\right)$$

$$y(t) = \tfrac{5}{3} + \tfrac{10}{3}\cos(\pi(t-0.1)) + \tfrac{10}{3}\cos(2\pi(t-0.1))$$

(d) For $y(t)$ to be constant, $Y(jw)$ should not have any components of the form $\delta(\omega-\omega_0)$. Therefore, only $\delta(\omega)$ must pass through the filter. So, we conclude that $\omega_{co} < \pi$.

In this case, $Y(jw)$ becomes

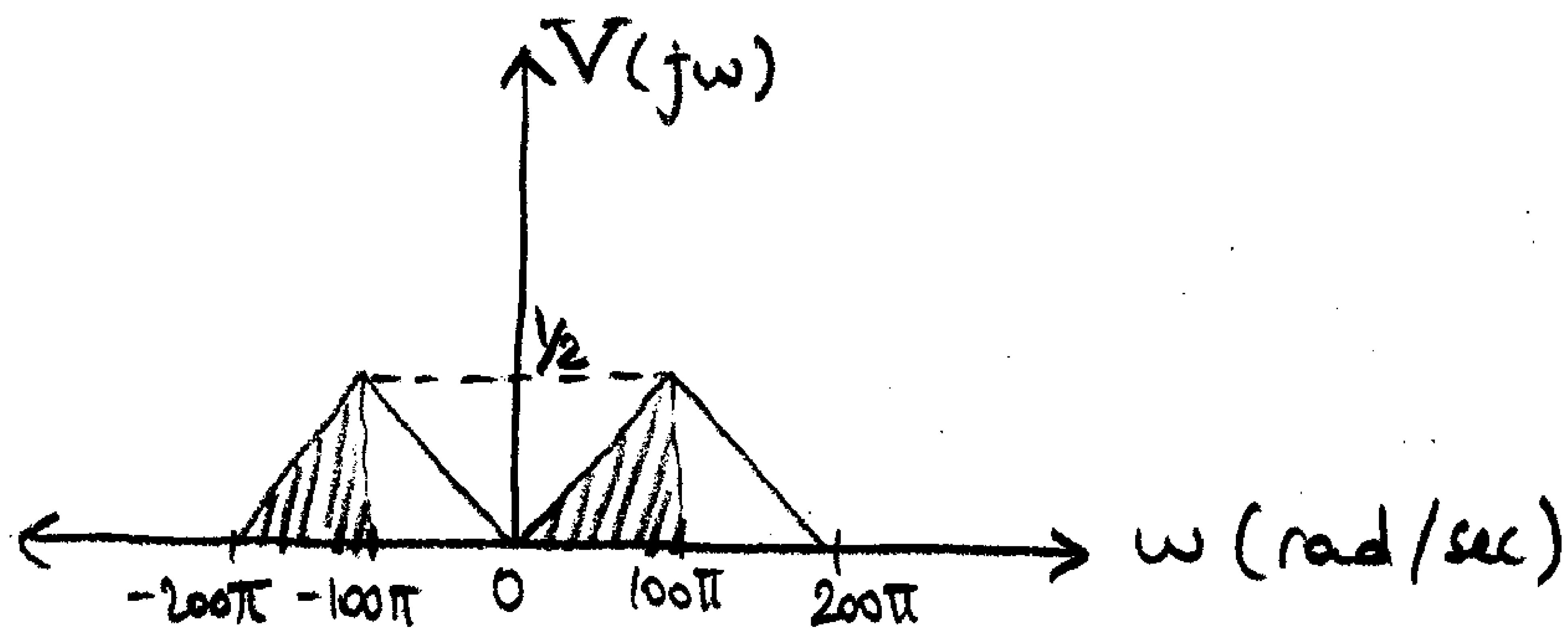
$$Y(jw) = \frac{5}{3} e^{-jw0.1} \cdot 2\pi\,\delta(w-0)$$

$$y(t) = \frac{5}{3} e^{j0} = \frac{5}{3} = C$$

14.4

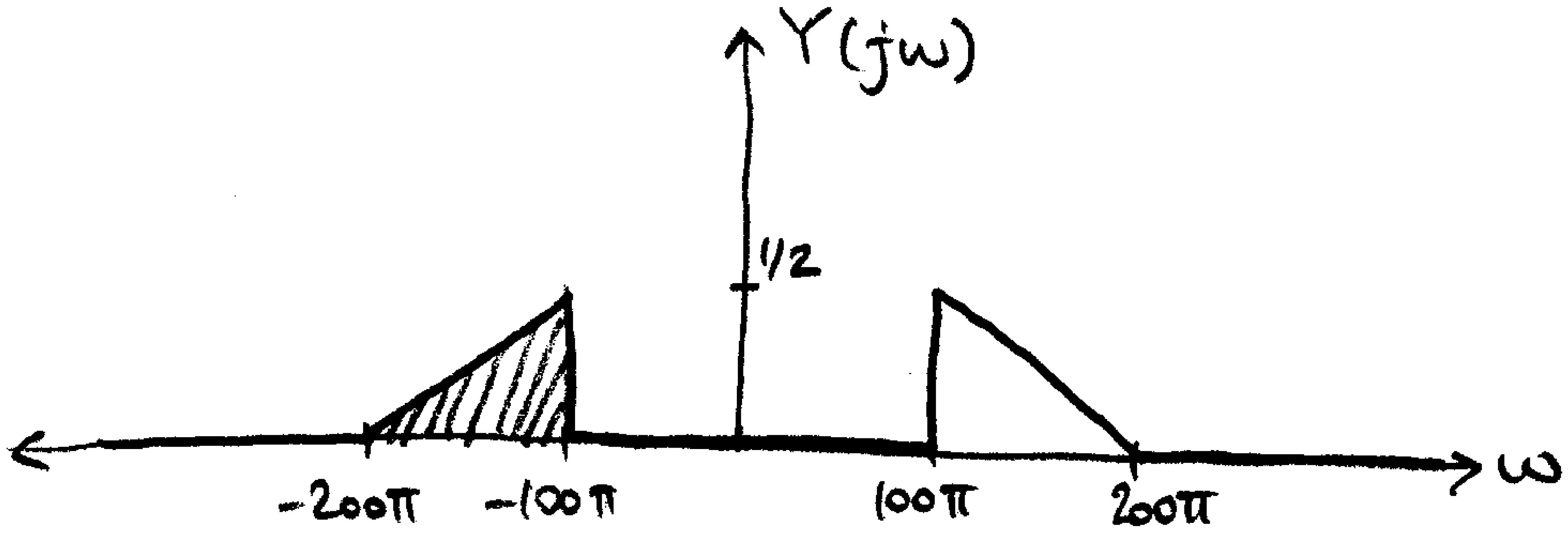
(a) $v(t) = x(t) \cdot \cos(100\pi t)$

$$V(j\omega) = \frac{1}{2} X(j(\omega - 100\pi)) + \frac{1}{2} X(j(\omega + 100\pi))$$

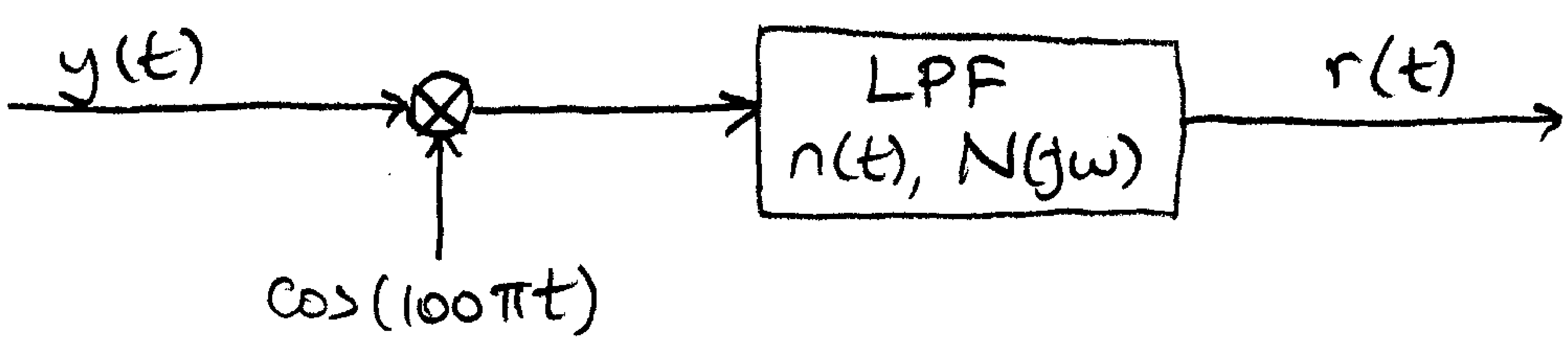


Note that $X(j\omega)$ is shifted left and right by 100π.

$H(j\omega)$ is a HPF with $\omega_{co} = 100\pi$. Therefore, it will remove all frequency components with $|\omega| < 100\pi$. Thus, we get



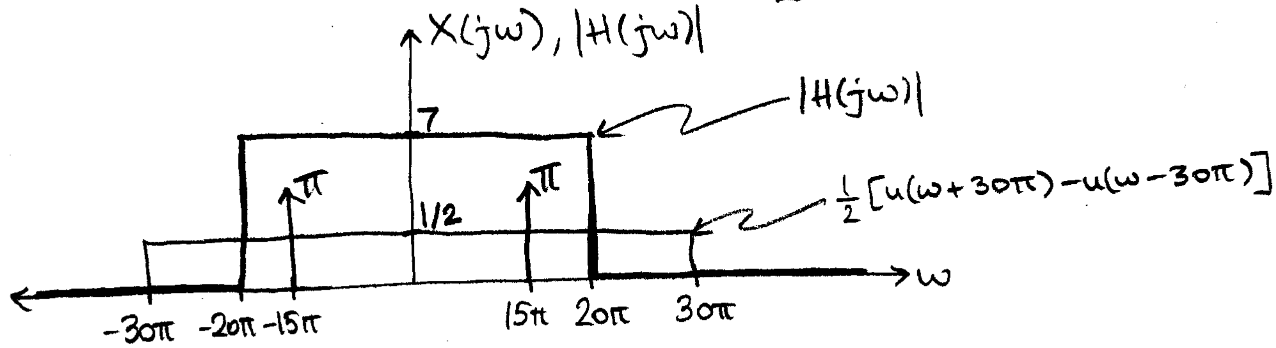
(b) If we take the plot above, shift it left and right by 100π and filter out all components with $|\omega| > 100\pi$, we will get the input $X(j\omega)$ back, i.e.,



where $N(j\omega) = \begin{cases} 0, & |\omega| > 100\pi \\ 4, & |\omega| \le 100\pi \end{cases}$

14.5

(a) $X(j\omega) = \pi\,\delta(\omega - 15\pi) + \pi\,\delta(\omega + 15\pi) + \frac{1}{2}\left[u(\omega + 30\pi) - u(\omega - 30\pi)\right]$



$$Y(j\omega) = \left(\pi\,\delta(\omega - 15\pi) + \pi\,\delta(\omega + 15\pi)\right) \cdot 7 \cdot e^{-j\frac{\omega}{25}}$$
$$+ \frac{1}{2}\left[u(\omega + 20\pi) - u(\omega - 20\pi)\right] \cdot 7 \cdot e^{-j\frac{\omega}{25}}$$

$$y(t) = 7 \cdot \cos\left(15\pi\left(t - 1/25\right)\right) + 7 \cdot \frac{\sin\left(20\pi\left(t - 1/25\right)\right)}{2\pi\left(t - 1/25\right)}$$

(b) $\cos(50\pi t) \xleftrightarrow{\mathcal{F}} \pi\,\delta(\omega - 50\pi) + \pi\,\delta(\omega + 50\pi)$

So, this component will be removed by the filter, and the output signal will be the same as in part (a)

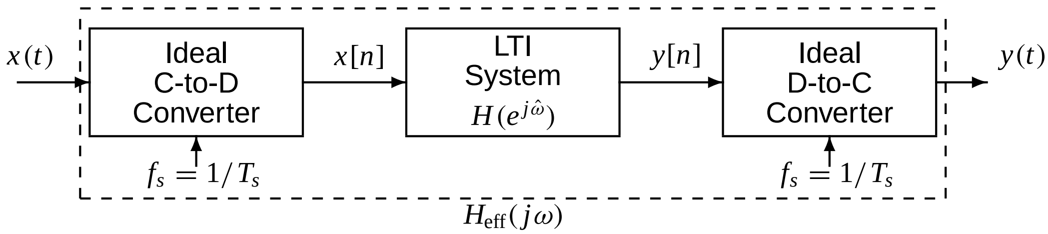
(c) $\frac{1}{2}\delta(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}$

So, this component will have the same effect as $\frac{\sin(30\pi t)}{2\pi t}$ after filtering because both components are equivalent to $\frac{1}{2}$ in the range $[-20\pi, 20\pi]$ in the frequency domain. Therefore, the output will be the same as in part (a).

(d) As explained above, the filtered output in part (a) is composed of two impulses at -15π and 15π, and a constant of $\frac{1}{2}$. In part (b), the third component is completely removed by the filter. In part (c), $\frac{1}{2}\delta(t)$ has the same net effect as $\frac{\sin(30\pi t)}{2\pi t}$ in the filtered range. Therefore, the output is the same in all parts.

PROBLEM 14.6:

Consider the following system for discrete-time filtering of a continuous-time signal:



(a) The difference equation

$$y[n] = 0.8y[n-1] + x[n] + x[n-2],$$

has a system function $H(z)$ and frequency response given by

$$H(z) = \frac{1+z^{-2}}{1-0.8z^{-1}} \qquad \text{and} \qquad H(e^{j\hat{\omega}}) = \frac{1+e^{-j2\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}$$

The overall effective frequency response, $H_{\text{eff}}(j\omega)$, of the above system is obtain by replacing $\hat{\omega}$ with $\omega/200$

$$H_{\text{eff}}(j\omega) = \begin{cases} \dfrac{1+e^{-j\omega/100}}{1-0.8e^{-j\omega/200}} & \text{for } |\omega| \le 200\pi \\ \text{not defined} & \text{for } |\omega| > 200\pi \end{cases}$$

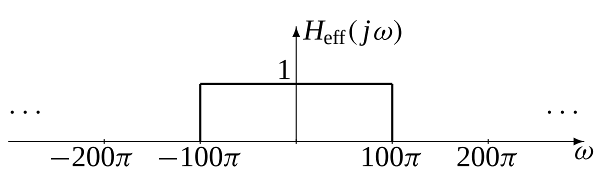
If the input is $x(t) = 2\cos(100\pi t)$, we evaluate $H_{\text{eff}}(j\omega)$ at $\omega = 100\pi$.

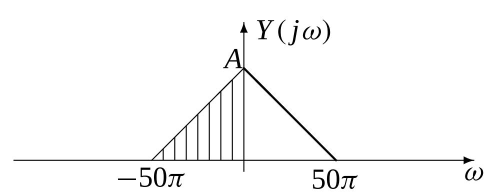
$$H_{\text{eff}}(j100\pi) = \frac{1+e^{-j100\pi/100}}{1-0.8e^{-j100\pi/200}} = \frac{1+e^{-j\pi}}{1-0.8e^{-j\pi}} = 0$$

So, the output is $y(t) = 0$.

(b) In order for the Fourier transforms of the input and output to satisfy the relation $Y(j\omega) = H_{\text{eff}}(j\omega)X(j\omega)$, the Sampling Theorem must be enforced so that we have no aliasing. Thus the *smallest* value of the sampling frequency f_s is $f_s = 50$ Hz, which is equivalent to $\omega_s = 100\pi$ rad/s.

(c) The discrete-time system's frequency response $H(e^{j\hat{\omega}})$ becomes the effective analog frequency response by using the change of variable $\hat{\omega} = \omega/f_s = \omega/200$. Thus we can use the frequency response plot of $H(e^{j\hat{\omega}})$ if we relabel the frequency axis.





The plot of $Y(j\omega)$, the Fourier transform of the output $y(t)$, will be identical to $X(j\omega)$ because the passband of the ideal LPF includes the frequency $\omega = 50\pi$ rad/s.

(d) If the input signal is going to pass through the lowpass filter unaltered, then the highest frequency $\omega = 50\pi$ rad/s must map to $\hat{\omega}$ that is less than or equal to $\hat{\omega} = \pi/2$ because $\pi/2$ is the edge of the passband of the LPF. In other words, we can write

$$\hat{\omega} = \omega/f_s \qquad \Rightarrow \qquad \frac{\pi}{2} = \frac{50\pi}{f_s} \qquad \Rightarrow \qquad f_s = 100 \text{ Hz}$$