# EE-2025 Fall-2003

## Lecture 6
## Fourier Series Analysis
## 8-Sept-2003

# General Info

- ***Help Sessions: M12, M6:15, T 7 and W6***
  - Every week
- ***Office Hours:*** Visit any Prof or TA
- ***Quiz #1 on 19-Sept***
  - Coverage: HW #1, #2, #3 and #4
- ***Bulletin Board: OFFICIAL ANNOUNCEMENTS***
  - Old Quizzes & Problems are linked via WebCT
- Prob Set #3 due This Week



# Lab Info

- Lab #2 Report
  - Turn in at the beginning of your lab time
    - Later on the same day? –5 points
  - Late? –10 points per day
  - Finish INSTRUCTOR VERIFICATION in Lab
    - Come prepared with some preliminary code
  - ERRATA ? ALWAYS Check the Bulletin Board
- Lab #3 has been posted
  - Bring Headphones to lab for the next few weeks
  - Read the Pre-Lab and do it before lab



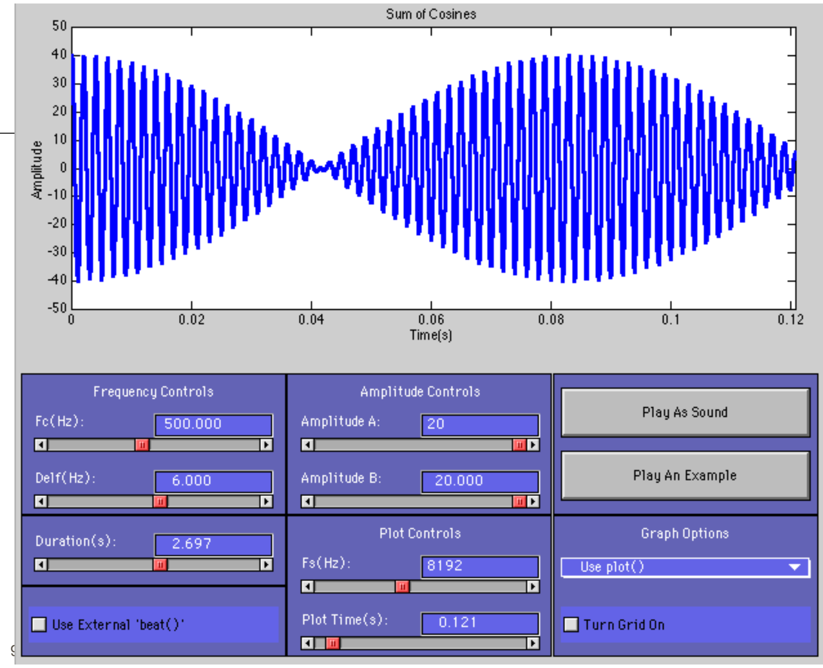
# The Rules

- Quizzes
  - NO make-ups given
  - Next Quiz would count for the one missed, IF excused
- Excused Absence
  - Must be written (by an “official”)
  - Notify ahead of time via e-mail
- Consult “INFO” on Web-CT for more details
  - Late Labs are –10 points per day
  - No late Homework

# FYI: Demo on CD-ROM

- Beat Control GUI
  - Called `beatcon.m`
  - Found in SP-First Toolbox
    - Install SP-First toolbox
  - Must be on MATLAB's path
    - Use the `path` or `addpath` commands






# INSTANTANEOUS FREQ of the Chirp

- Chirp Signals have Quadratic phase
- Freq will change LINEARLY vs. time

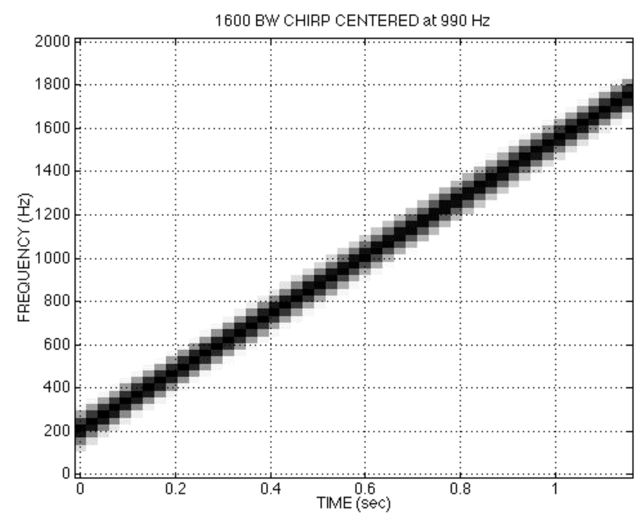
$$x(t) = A\cos(\alpha t^2 + \beta t + \varphi)$$

$$\Rightarrow \psi(t) = \alpha t^2 + \beta t + \varphi$$

$$\Rightarrow \omega_i(t) = \frac{d}{dt}\psi(t) = 2\alpha t + \beta$$



# CHIRP SPECTROGRAM



## Question: Create a Chirp

- Chirp should start at 200 Hz and end at 3200 Hz, and last for 1.5 sec.

$$x(t) = A\cos(\alpha t^2 + \beta t + \varphi)$$

$$\Rightarrow \alpha = ?$$

$$\Rightarrow \beta = ?$$

$$\Rightarrow \varphi = ?$$

Lecture



## READING ASSIGNMENTS

- This Lecture:
  - **Fourier Series in Ch 3, Sects 3-4, 3-5 & 3-6**
    - Replaces pp. 62-66 in Ch 3 in DSP First
    - Notation: $a_k$ for Fourier Series
- Other Reading:
  - Next Lecture: More Fourier Series



## LECTURE OBJECTIVES

- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi k/T_0)t}dt$$

- **ANALYSIS** via Fourier Series
  - For **PERIODIC** signals: $\mathbf{x(t+T_0) = x(t)}$
  - Later: spectrum from the Fourier Series



## HISTORY

- Jean Baptiste Joseph Fourier
  - 1807 thesis (memoir)
    - On the Propagation of Heat in Solid Bodies
  - Heat !
  - Napoleonic era
- http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html

Joseph Fourier

lived from 1768 to 1830

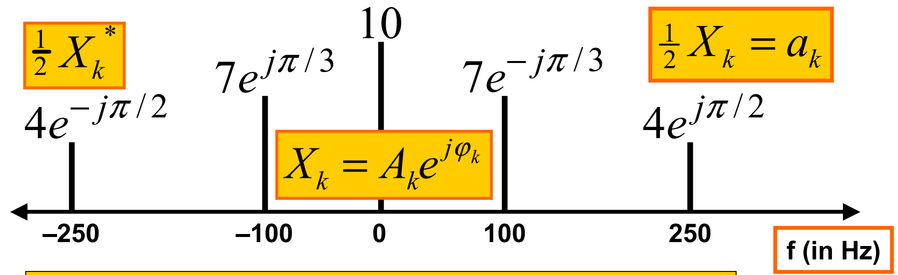
**Fourier** studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

*Find out more at:*
http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fourier.html



# SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq




$$x(t) = a_0 + \sum_{k=1}^{N} \left\{ a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \right\}$$



# Harmonic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$



# Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

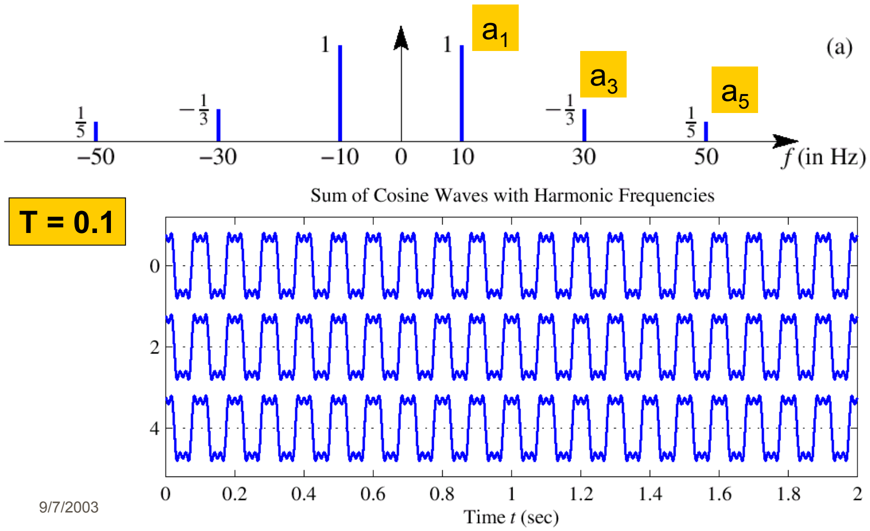
$$a_k = \tfrac{1}{2} X_k = \tfrac{1}{2} A_k e^{j\varphi_k}$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$X_k = A_k e^{j\varphi_k}$$

COMPLEX AMPLITUDE

# Harmonic Signal (3 Freqs)






# SYNTHESIS vs. ANALYSIS

- SYNTHESIS
  - **Easy**
  - Given $(\omega_k, A_k, \phi_k)$ create x(t)
- Synthesis can be HARD
  - Synthesize Speech so that it sounds good
- ANALYSIS
  - **Hard**
  - Given x(t), extract $(\omega_k, A_k, \phi_k)$
  - How many?
  - Need algorithm for computer



# STRATEGY: x(t) → $a_k$

- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 k t}dt$$



# INTEGRAL Property of exp(j)

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}dt = \frac{T_0}{-j2\pi m}e^{-j(2\pi/T_0)mt}\Big|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m}(e^{-j2\pi m}-1)$$

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}dt = 0$$

$$m \neq 0\,)$$

$$\omega_0 = \frac{2\pi}{T_0}$$

## ORTHOGONALITY of exp(j)

- PRODUCT of exp(+j ) and exp(-j )

$$\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)\ell t} e^{-j(2\pi/T_0)kt}\,dt = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$$

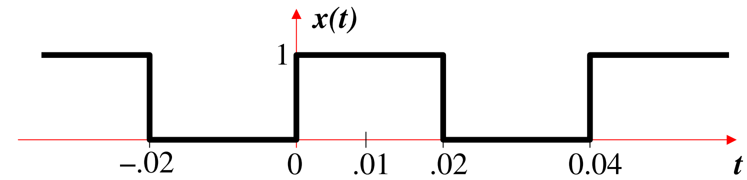
$$\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)(\ell - k)t}\,dt$$



## SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \tfrac{1}{2}T_0 \\ 0 & \tfrac{1}{2}T_0 \le t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.






## FS for a SQUARE WAVE {a$_k$}

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt}\,dt \qquad (k \neq 0)$$

$$a_k = \frac{1}{.04}\int_0^{.02} 1\, e^{-j(2\pi/.04)kt}\,dt = \frac{1}{.04(-j2\pi k/.04)} e^{-j(2\pi/.04)kt}\Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)}\left(e^{-j(\pi)k} - 1\right) = \frac{1-(-1)^k}{j2\pi k}$$



## DC Coefficient: $a_0$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt}\,dt \qquad (k = 0)$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0}(\text{Area})$$

$$a_0 = \frac{1}{.04}\int_0^{.02} 1\,dt = \frac{1}{.04}(.02 - 0) = \tfrac{1}{2}$$

# Fourier Coefficients $a_k$

- $a_k$ is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, $T_0$

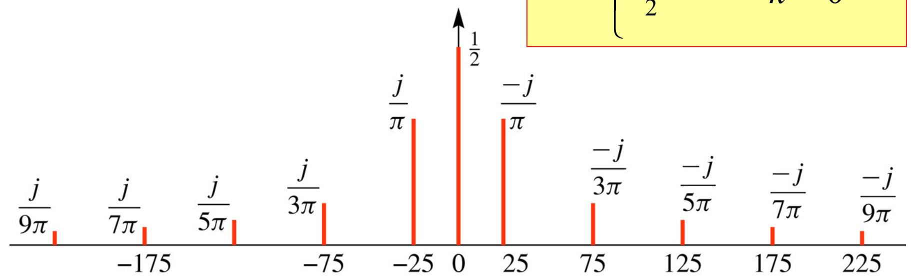
$$a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$



# Spectrum from Fourier Series

$$\omega_0 = 2\pi/(0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \frac{-j}{\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$






# Fourier Series Integral

- HOW do you determine $a_k$ from x(t) ?

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

Fundamental Frequency $f_0 = 1/T_0$

$a_{-k} = a_k^*$ when $x(t)$ is real

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt \qquad \text{(DC component)}$$