ECE-2025 Fall-2003

Lecture 17

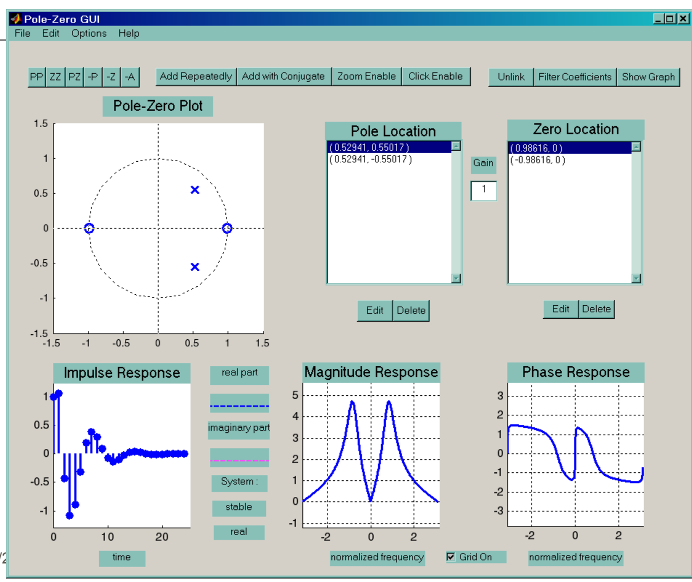
H(z) & Frequency Response for IIR Systems

27-Oct-03

Info: Web-CT, Lab, HW

- Quiz #2: Results
  - Median = 68
  - Quiz #2: Resolve any grade changes by Friday (7-Nov)
  - Quiz #3 is 21-Nov (Friday)
- Prob Set #8 is due this week
- Lab #9 is DTMF (Touch-Tone) Lab
  - Formal Report

PeZ Demo: Pole-Zero Placing



Lecture

READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, Sects. 8-4 8-5 & 8-6
- Other Reading:
  - Recitation: Chapter 8, all
    - POLE-ZERO PLOTS

LECTURE OBJECTIVES

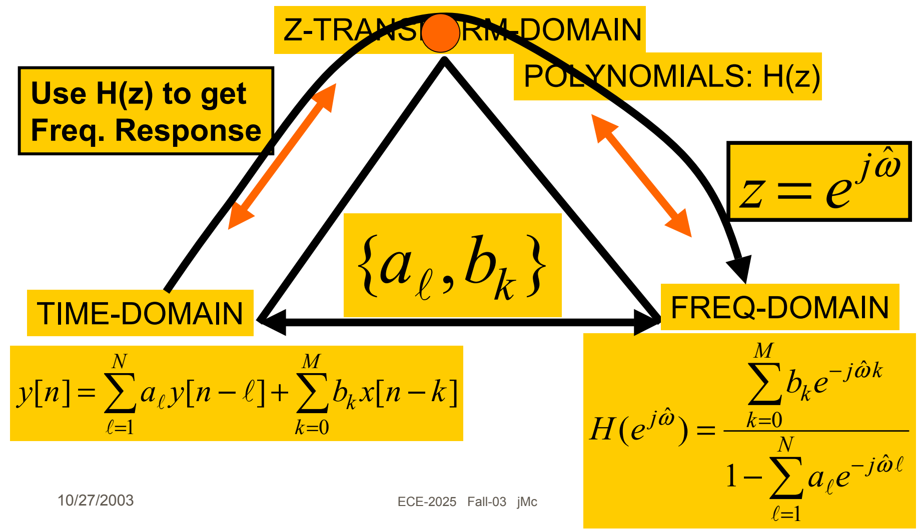
- SYSTEM FUNCTION: H(z)
- H(z) has **POLES** and ZEROS
- FREQUENCY RESPONSE of IIR
  - Get H(z) first

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

THREE DOMAINS



H(z) = z-Transform{ h[n] }

- FIRST-ORDER IIR FILTER:

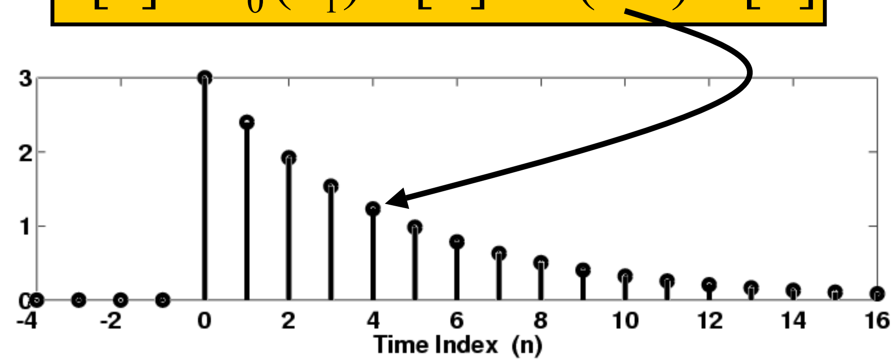
$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$h[n] = b_0 (a_1)^n u[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

TYPICAL IMPULSE RESPONSE

$$h[n] = b_0 (a_1)^n u[n] = 3(0.8)^n u[n]$$



DELAY PROPERTY of X(z)

- DELAY in TIME<-->Multiply X(z) by z^{-1}

$$x[n] \leftrightarrow X(z)$$

$$x[n-1] \leftrightarrow z^{-1}X(z)$$

Proof: $$\sum_{n=-\infty}^{\infty} x[n-1]z^{-n} = \sum_{\ell=-\infty}^{\infty} x[\ell]z^{-(\ell+1)}$$

$$= z^{-1}\sum_{\ell=-\infty}^{\infty} x[\ell]z^{-\ell} = z^{-1}X(z)$$

Z-TRANSFORM TABLES

SHORT TABLE OF z-TRANSFORMS

	$x[n]$	$\Longleftrightarrow$	$X(z)$
1.	$ax_1[n] + bx_2[n]$	$\Longleftrightarrow$	$aX_1(z) + bX_2(z)$
2.	$x[n-n_0]$	$\Longleftrightarrow$	$z^{-n_0}X(z)$
3.	$y[n] = x[n] * h[n]$	$\Longleftrightarrow$	$Y(z) = H(z)X(z)$
4.	$\delta[n]$	$\Longleftrightarrow$	1
5.	$\delta[n-n_0]$	$\Longleftrightarrow$	z^{-n_0}
6.	$a^n u[n]$	$\Longleftrightarrow$	$\dfrac{1}{1-az^{-1}}$

SYSTEM FUNCTION of IIR

- DERIVE the FILTER COEFFICIENTS

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)$$

$$(1 - a_1 z^{-1})Y(z) = (b_0 + b_1 z^{-1})X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{B(z)}{A(z)}$$

SYSTEM FUNCTION

- DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]$$

- **READ** the FILTER COEFFS: H(z)

$$Y(z) = \left(\frac{3 - 2z^{-1}}{1 - 0.8z^{-1}}\right)X(z)$$

POLES & ZEROS

- ROOTS of Numerator & Denominator

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} \rightarrow H(z) = \frac{b_0 z + b_1}{z - a_1}$$

$$b_0 z + b_1 = 0 \quad \Rightarrow z = -\frac{b_1}{b_0}$$

ZERO: H(z)=0

$$z - a_1 = 0 \quad \Rightarrow z = a_1$$

POLE: H(z) → inf

EXAMPLE: Poles & Zeros

- VALUE of H(z) at POLES is **INFINITE**

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

$$H(z) = \frac{2 + 2(-1)}{1 - 0.8(-1)} = 0$$

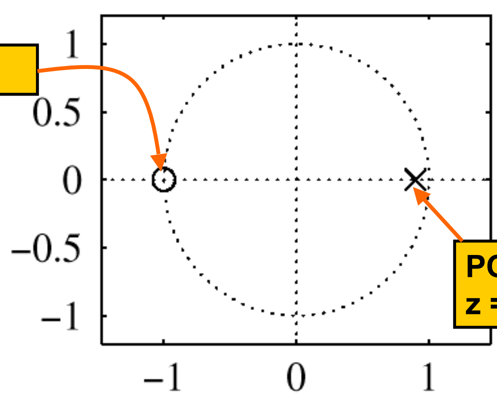
ZERO at z= -1

$$H(z) = \frac{2 + 2(\frac{4}{5})^{-1}}{1 - 0.8(\frac{4}{5})^{-1}} = \frac{\frac{9}{2}}{0} \rightarrow \infty$$

POLE at z=0.8

POLE-ZERO PLOT

ZERO at z = -1



POLE at z = 0.8

$$\frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

FREQUENCY RESPONSE

- SYSTEM FUNCTION: H(z)
- H(z) has DENOMINATOR
- FREQUENCY RESPONSE of IIR
  - We have H(z)

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z = e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

FREQUENCY RESPONSE

- EVALUATE on the UNIT CIRCLE

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}} = \frac{b_0 + b_1 e^{-j\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}}}$$

FREQ. RESPONSE FORMULA

$$H(z) = \frac{2+2z^{-1}}{1-0.8z^{-1}} \rightarrow H(e^{j\hat{\omega}}) = \frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}$$

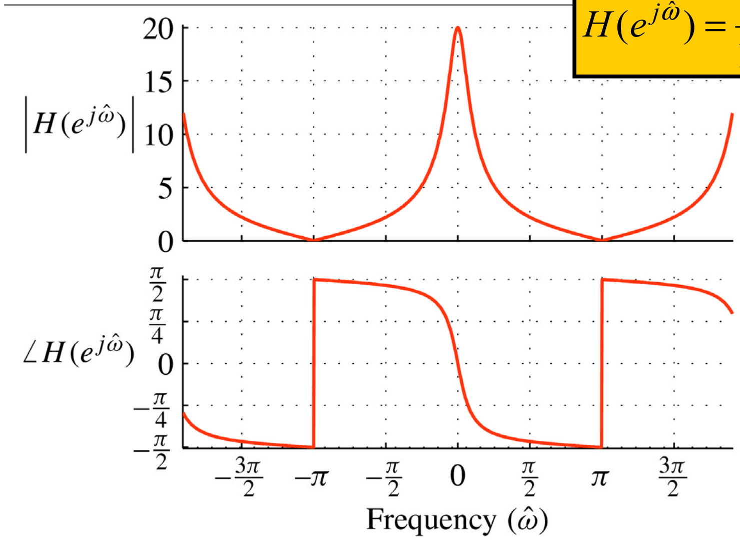
$$\left|H(e^{j\hat{\omega}})\right|^2 = \left|\frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}\right|^2 = \frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}} \cdot \frac{2+2e^{j\hat{\omega}}}{1-0.8e^{j\hat{\omega}}}$$

$$\frac{4+4+4e^{-j\hat{\omega}}+4e^{j\hat{\omega}}}{1+0.64-0.8e^{-j\hat{\omega}}-0.8e^{j\hat{\omega}}} = \frac{8+8\cos\hat{\omega}}{1.64-1.6\cos\hat{\omega}}$$

$$@\,\hat{\omega}=0, \quad \left|H(e^{j\hat{\omega}})\right|^2 = \frac{8+8}{0.04} = 400, \quad @\,\hat{\omega}=\pi\,?$$

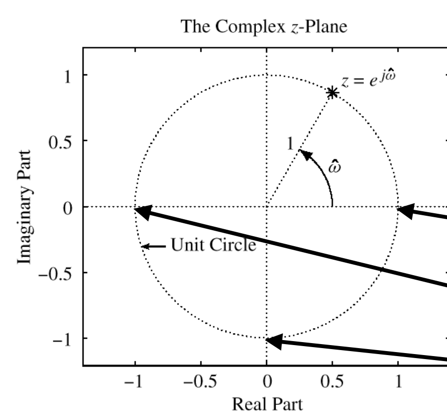
Frequency Response Plot

$$H(e^{j\hat{\omega}}) = \frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}$$



UNIT CIRCLE

- MAPPING BETWEEN z and $\hat{\omega}$



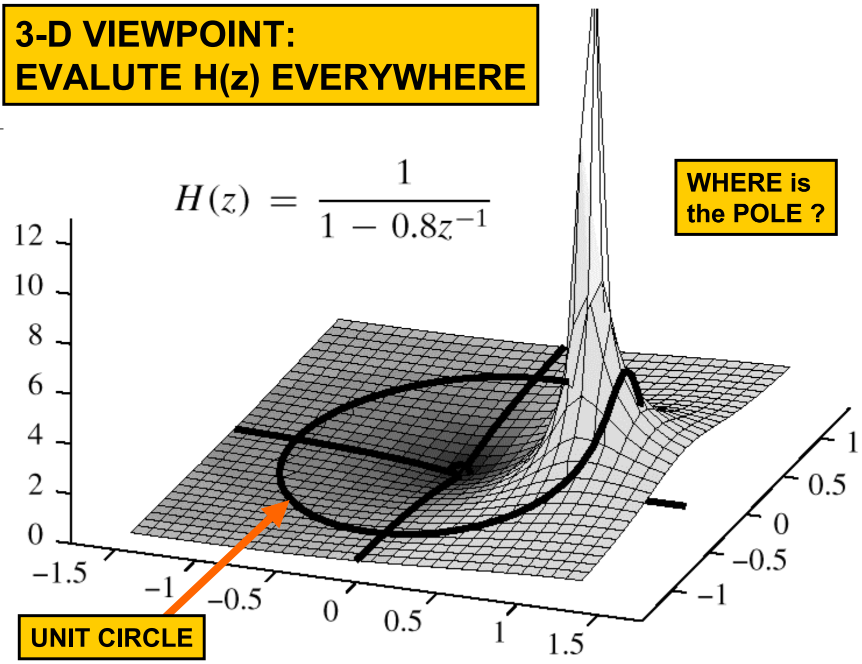
$$z = e^{j\hat{\omega}}$$

$$z = 1 \leftrightarrow \hat{\omega} = 0$$

$$z = -1 \leftrightarrow \hat{\omega} = \pm\pi$$

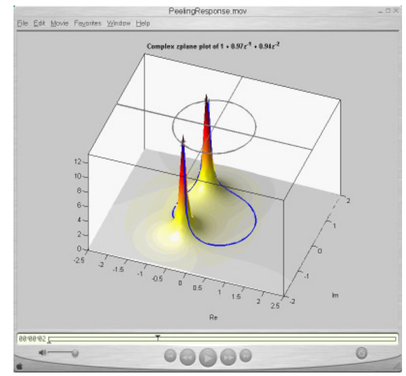
$$z = \pm j \leftrightarrow \hat{\omega} = \pm\tfrac{1}{2}\pi$$

3-D VIEWPOINT: EVALUTE H(z) EVERYWHERE

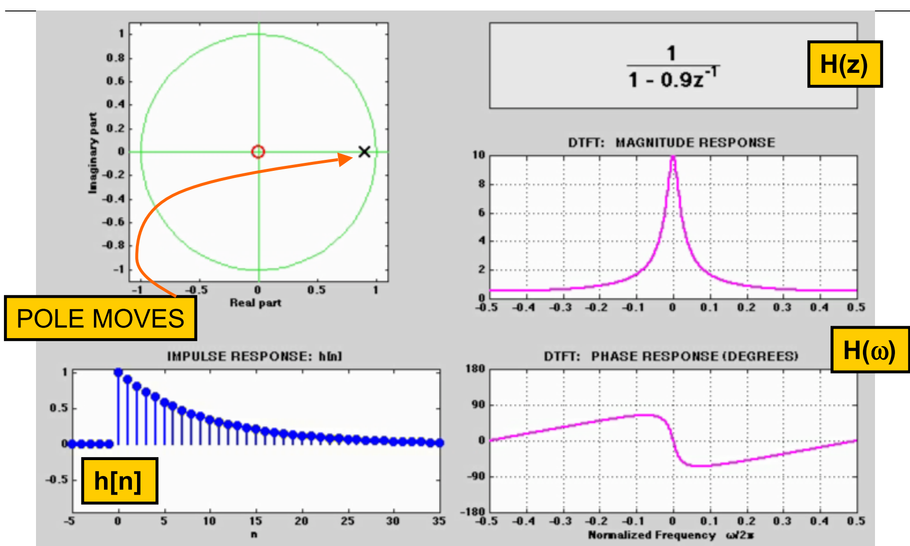


MOVIE for H(z) in 3-D

- POLES to H(z) to Frequency Reponse
  - TWO POLES SHOWN



3 DOMAINS MOVIE: IIR



SECOND-ORDER FILTERS

- Two FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2]$$
$$+ b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

MORE POLES

- Denominator is QUADRATIC
  - 2 Poles: REAL
  - or COMPLEX CONJUGATES

$$\frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 - a_1 z - a_2}$$

PROPERTY OF REAL POLYNOMIALS

A polynomial of degree N has N roots. If all the coefficients of the polynomial are real, the roots either must be real, or must occur in complex conjugate pairs.

TWO COMPLEX POLES

- Find Impulse Response ?
  - Can OSCILLATE vs. n
  - "RESONANCE"

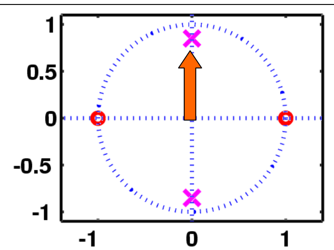
$$(p_k)^n = (re^{j\theta})^n = r^n e^{jn\theta}$$

- Find **FREQUENCY RESPONSE**
  - Depends on Pole Location
  - Close to the Unit Circle?
    - Make BANDPASS FILTER

$$\text{pole} = re^{j\theta}$$

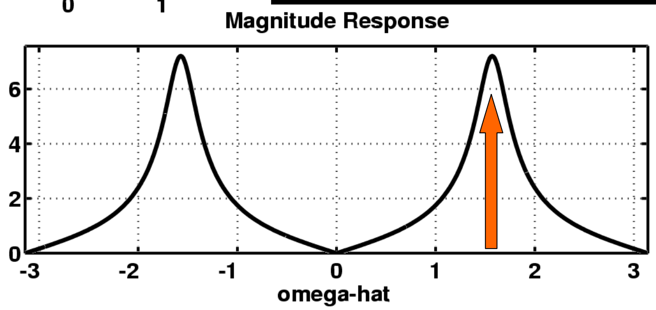
$$r \to 1\,?$$

FREQUENCY RESPONSE from POLE-ZERO PLOT



$$\frac{1 - z^{-2}}{1 + 0.7225 z^{-2}}$$

$$H(e^{j\hat{\omega}}) = \frac{1 - e^{-j2\hat{\omega}}}{1 + 0.7225 e^{-j2\hat{\omega}}}$$



2nd ORDER EXAMPLE

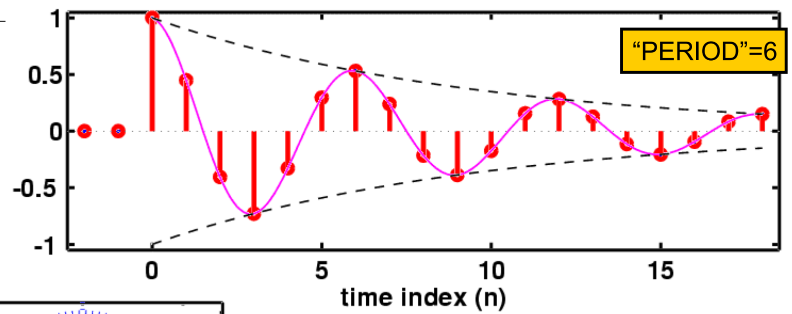
$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3} n) u[n] = (0.9)^n \tfrac{1}{2}(e^{j\pi n/3} + e^{-j\pi n/3}) u[n]$$

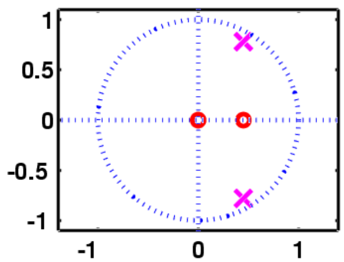
$$H(z) = \frac{0.5}{1 - 0.9 e^{j\pi/3} z^{-1}} + \frac{0.5}{1 - 0.9 e^{-j\pi/3} z^{-1}}$$

$$H(z) = \frac{1 - 0.9\cos(\frac{\pi}{3}) z^{-1}}{(1 - 0.9 e^{j\pi/3} z^{-1})(1 - 0.9 e^{-j\pi/3} z^{-1})}$$

$$H(z) = \frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

h[n]: Decays & Oscillates

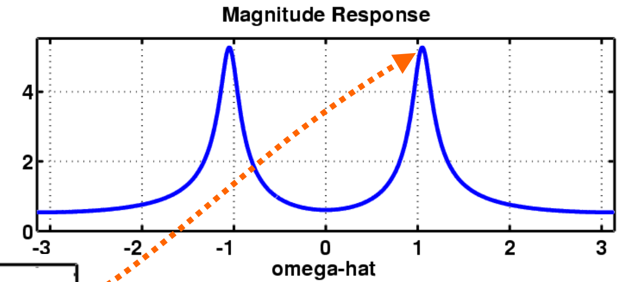


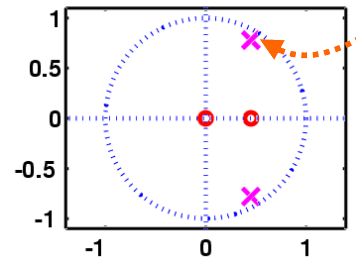


$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3} n) u[n]$$

$$\frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

Complex POLE-ZERO PLOT





$$\frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

2nd ORDER EX: n-Domain

$$\frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

$$y[n] = 0.9 y[n-1] - 0.81 y[n-2] + x[n] - 0.45 x[n-1]$$

```
aa = [ 1, -0.9, 0.81 ];
bb = [ 1, -0.45 ];
nn = -2:19;
hh = filter( bb, aa, (nn==0) );
HH = freqz( bb, aa, [-pi,pi/100:pi] );
```


2nd ORDER Z-transform PAIR

GENERAL ENTRY for z-Transform TABLE

$$h[n] = r^n \cos(\theta n) u[n]$$

$$H(z) = \frac{1 - r\cos\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

$$h[n] = A r^n \cos(\theta n + \varphi) u[n]$$

$$H(z) = A \frac{\cos\varphi - r\cos(\theta - \varphi) z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$