PROBLEM SET #4 : Solution

(4.1) $s(t) = \cos\left(2\pi f_c t - 2\pi f_c \frac{2}{c}\sqrt{x_0^2 + (y_0 + vt)^2}\right)$

This is of the form $s(t) = \cos(\Psi(t))$, where the "angle" function $\Psi(t)$ is more complicated than our usual $(\omega_0 t + \phi)$.

(a) From p. 61, EQN. (3.46), the "instantaneous frequency" is

$$f_i(t) = \frac{1}{2\pi}\frac{d}{dt}\Psi(t) = \frac{1}{2\pi}\frac{d}{dt}\left\{2\pi f_c t - 2\pi f_c \frac{2}{c}\sqrt{x_0^2 + (y_0 - vt)^2}\right\}$$

$$= f_c - \frac{2f_c}{c}\cdot\frac{d}{dt}\left(x_0^2 + (y_0 - vt)^2\right)^{1/2}$$

$$= f_c - \frac{2f_c}{c}\cdot\frac{1}{2}\left(x_0^2 + (y_0 - vt)^2\right)^{-1/2}\cdot 2(y_0 - vt)\cdot(-v)$$

$$= \boxed{f_c + \frac{2vf_c}{c}\frac{(y_0 - vt)}{\sqrt{x_0^2 + (y_0 - vt)^2}}}$$

(see the next page to see what this looks like)

(b) With $y_0 = 0$, $f_i(0) = ?$

$$f_i(0) = f_c + \frac{2vf_c}{c}\frac{(0-0)}{\sqrt{x_0^2 + (0-0)^2}} = \boxed{f_c}$$

prepared by M. Richards
9/9/03

ECE 2025
Problem Set #4 solution
Page 1a

The waveform used in problem 4.1 is the model used, among other things, in *synthetic aperture radar* (SAR), or "imaging radar". Some typical parameters for this waveform might be:

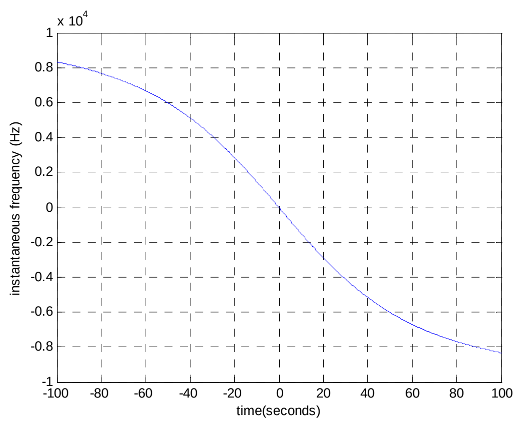
- f_c = 10 GHz = 10^{10} Hz;
- v = 150 m/s (velocity of the airplane the radar is on);
- c = 3×10^8 m/s (speed of light);
- x_0 = 10 km (range to the target to be imaged); and
- y_0 = 0 (target is abreast of the radar at time t = 0.

and we have just seen that the instantaneous frequency of this waveform is

$$f_i(t) = f_c + \frac{2vf_c}{c}\frac{(y_0 - vt)}{\sqrt{x_0^2 + (y_0 - vt)^2}}$$

The numerical values of the term $\frac{2vf_c}{c}\frac{(y_0 - vt)}{\sqrt{x_0^2 + (y_0 - vt)^2}}$ are usually small compared to f_c.

To see what $f_i(t)$ looks like, I've plotted below only $f_i(t)$-f_c, *i.e.* just the time-varying frequency term immediately above, for these parameters. I've plotted this for $-100 \le t \le +100$ seconds, which is much more time than we would typically track it; 1 or 2 seconds would be more likely than 200 seconds, and over that region it looks very close to linear.

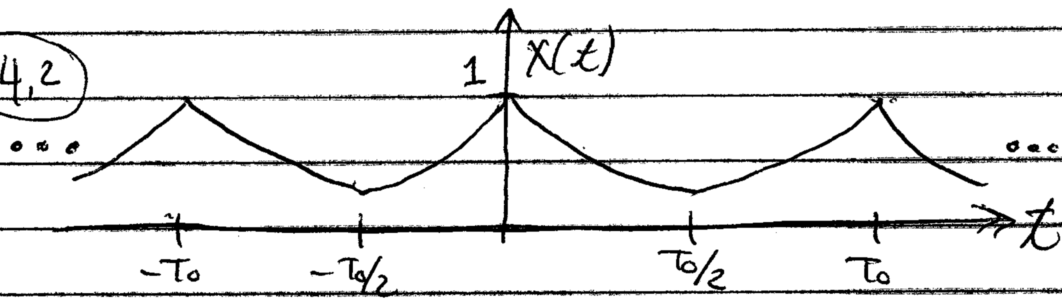


And if you've never seen a SAR image, there are some good ones at

http://www.sandia.gov/radar/sar.html

I'd include one here, but it would make this file too big.

4,2



$$x(t) = e^{-a|t|}, \quad |t| \le T_0/2 \; ; \text{ repeats periodically}$$

Alternatively,

$$x(t) = \begin{cases} e^{-at}, & 0 < t \le T_0/2 \\ e^{+at}, & -T_0/2 \le t \le 0 \\ \text{repeats w/ period } T_0 \end{cases}$$

To get the coefficients:

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t)\, e^{-j\frac{2\pi}{T_0}kt}\, dt$$

But since the integrand is periodic with period T_0, we can integrate over any interval of length T_0 in t and we will get the same "area" (integral); in this case it is more convenient to integrate from $-T_0/2$ to $+T_0/2$:

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} e^{-a|t|}\, e^{-j\frac{2\pi}{T_0}kt}\, dt$$

$$= \frac{1}{T_0} \int_{-T_0/2}^{0} e^{+at}\, e^{-j\frac{2\pi}{T_0}kt}\, dt + \frac{1}{T_0} \int_0^{T_0/2} e^{-at}\, e^{-j\frac{2\pi}{T_0}kt}\, dt$$

Also, let's use $\omega_0 = 2\pi f_0 = 2\pi/T_0$ to simplify a little.
Now we have:

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{0} e^{(a-j\omega_0 k)t}\,dt + \frac{1}{T_0}\int_0^{T_0/2} e^{-(a+j\omega_0 k)t}\,dt$$

Do the integrals:

$$a_k = \frac{1}{T_0}\frac{1}{(a-j\omega_0 k)} e^{(a-j\omega_0 k)t}\Big|_{-T_0/2}^{0}$$

$$+ \frac{1}{T_0}\frac{-1}{(a+j\omega_0 k)} e^{-(a+j\omega_0 k)t}\Big|_{0}^{T_0/2}$$

$$= \frac{1}{T_0}\frac{1}{a-j\omega_0 k}\left[1-e^{-(a-j\omega_0 k)T_0/2}\right]$$

$$+ \frac{1}{T_0}\frac{1}{(a+j\omega_0 k)}\left[1-e^{-(a+j\omega_0 k)T_0/2}\right]$$

cross-multiply the $(a-j\omega_0 k)$ and $(a+j\omega_0 k)$ to get a common constant out front

$$a_k = \frac{1}{T_0(a^2+\omega_0^2k^2)}\Big\{(a+j\omega_0 k)\left[1-e^{-(a-j\omega_0 k)T_0/2}\right]$$

$$+ (a-j\omega_0 k)\left[1-e^{-(a+j\omega_0 k)T_0/2}\right]\Big\}$$

$$= \frac{1}{T_0(a^2+\omega_0^2k^2)}\Big\{a+\cancel{j\omega_0 k} - a e^{-aT_0/2}e^{+j\omega_0 kT_0/2}$$

$$-j\omega_0 k e^{-aT_0/2}e^{+j\omega_0 kT_0/2} + a - \cancel{j\omega_0 k}$$

$$- a e^{-aT_0/2}e^{-j\omega_0 kT_0/2} + j\omega_0 k e^{-aT_0/2}e^{-j\omega_0 kT_0/2}\Big\}$$

$$a_k = \frac{1}{T_0(a^2+\omega_0^2k^2)}\left\{2a - ae^{-aT_0/2}\left[e^{j\omega_0 k T_0/2} + e^{-j\omega_0 k T_0/2}\right]\right.$$
$$\left. -j\omega_0 k\left[e^{-aT_0/2}\left[e^{j\omega_0 k T_0/2} - e^{-j\omega_0 k T_0/2}\right]\right]\right\}$$

The inverse Euler's formulas are

$$\cos\theta = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]; \quad \sin\theta = \frac{1}{2j}\left[e^{j\theta} - e^{-j\theta}\right]$$

Be on the lookout for opportunities to use these! Here we have

$$a_k = \frac{1}{T_0(a^2+\omega_0^2k^2)}\left\{2a - 2ae^{-aT_0/2}\cos(\omega_0 k T_0/2)\right.$$
$$\left. -j\omega_0 k(2j)\, e^{-aT_0/2}\sin(\omega_0 k T_0/2)\right\}$$

Finally,

$$\boxed{a_k = \frac{2}{T_0(a^2+\omega_0^2k^2)}\left\{a - ae^{-aT_0/2}\cos\left(\frac{\omega_0 k T_0}{2}\right) + 2\omega_0 k e^{-aT_0/2}\sin\left(\frac{\omega_0 k T_0}{2}\right)\right\}}$$

4.3 A ~~xxx~~ signal composed of a sum of sinusoids (and maybe a DC component) is periodic if the frequencies of the non-DC terms are all multiples of some fundamental frequency f_0, e.g. $f_1 = k_1 f_0$, $f_2 = k_2 f_0$, etc. This also means that the ratio of any 2 of the non-zero frequencies is a rational number, e.g. $f_1/f_2 = k_1 f_0 / k_2 f_0 = k_1/k_2$ (a rational number).

(a) $x(t) = 7 + \sin(1999\pi t - \pi/2) + 2\cos(2000\pi t + \pi/3)$

The nonzero frequencies are $f_1 = \frac{1999\pi}{2\pi}$ Hz $= 999.5$ Hz, $f_2 = \frac{2000\pi}{2\pi}$ Hz $= 1000$ Hz

$\frac{f_2}{f_1} = \frac{2000}{1999}$ This is rational, so $\boxed{\text{the signal is periodic}}$

The fundamental frequency is the gcd of 999.5 & 1000, which is 0.5 Hz! Thus the fundamental period is

$$\boxed{T_0 = \frac{1}{f_0} = 2 \text{ seconds}}$$

The nonzero terms ~~xxx~~ of the Fourier series are obtained by inspection:

$$x(t) = 7 + \frac{1}{2j}\left[e^{j(1999\pi t - \pi/2)} - e^{-j(1999\pi t - \pi/2)}\right]$$

$$+ 2\cdot\frac{1}{2}\left[e^{j(2000\pi t + \pi/3)} + e^{-j(2000\pi t + \pi/3)}\right]$$

since $\frac{1}{j} = -j = e^{-j\pi/2}$, we get

~~x(t) = 7~~

$$x(t) = 7 + \frac{1}{2}\overset{-1}{\cancel{e^{-j\pi}}}\, e^{j1999\pi t} - \frac{1}{2}\overset{1}{\cancel{e^{j(0)}}}\, e^{-j1999\pi t}$$

$$+ e^{j\pi/3} e^{j2000\pi t} + e^{-j\pi/3} e^{-j2000\pi t}$$

By inspection, the nonzero coefficients are

$$\boxed{a_0 = 7 \quad a_{1999} = -\frac{1}{2}, \quad a_{-1999} = -1/2, \quad a_{2000} = e^{j\pi/3}, \quad a_{-2000} = e^{-j\pi/3}}$$

(Note that since this is a real-valued signal, $a_k = a_{-k}^*$)

(b) $x(t) = \cos(\sqrt{3}\pi t + \pi/2) + \cos(\pi t - 4\pi/7)$

$$f_1 = \frac{\sqrt{3}\pi}{2\pi} = \frac{\sqrt{3}}{2}, \quad f_2 = \frac{1}{2} \Rightarrow \text{ratio} = \frac{f_1}{f_2} = \sqrt{3}$$

This is an irrational number, so the signal is $\boxed{\text{not periodic}}$

(c) $x(t) = \displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{|k|^3 + 2} e^{j60\pi k t}$

This is already in the form of a Fourier series, so the signal is periodic. The frequencies are of the form $60\pi k = \omega_k \Rightarrow f_k = 30k$ Hz $\Rightarrow f_0 = 30$ Hz

$\Rightarrow \boxed{T_0 = \frac{1}{30} \text{ secs}}$ and, by inspection $\boxed{a_k = \frac{1}{|k|^3 + 2}}$

4.4 $x(t) = 4\cos(250\pi t + \pi/3)\cos(1000\pi t)$

Use Euler's formulae to write

$$x(t) = 4 \cdot \frac{1}{2}\left[e^{j(250\pi t + \pi/3)} + e^{-j(250\pi t + \pi/3)}\right] \cdot \frac{1}{2}\left[e^{j1000\pi t} + e^{-j1000\pi t}\right]$$

$$= e^{j\pi/3} e^{j1250\pi t} + e^{j\pi/3} e^{-j750\pi t} + e^{-j\pi/3} e^{+j750\pi t} + e^{-j\pi/3} e^{-j1250\pi t}$$

We see that frequencies of $\pm 1250\pi$ rad/s and $\pm 750\pi$ rad/sec (or ± 625 Hz and ± 375 Hz) are present. The ratio of these is a rational number, so the $\boxed{\text{signal is periodic}}$. The gcd of 375 and 625 is 125 Hz, so $f_0 = 125$ Hz. The fundamental period is

$$\boxed{T_0 = \frac{1}{f_0} = \frac{1}{125} \text{ secs.}}$$

The non-zero Fourier series coefficients are

$$\boxed{\begin{aligned} a_3 &= e^{-j\pi/3}, \quad a_{-3} = e^{+j\pi/3} \\ a_5 &= e^{j\pi/3}, \quad a_{-5} = e^{-j\pi/3} \end{aligned}}$$

4.5 From Fall 2001, problem set 4, problem 4.4:

Starting point: Fourier series: $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

(a) $y(t) = A\,x(t) = A \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

$$= \sum_{k=-\infty}^{\infty} (A a_k) e^{jk\omega_0 t}$$

this is the form of a Fourier series with coefficients $\boxed{b_k = A a_k}$.

(b) $y(t) = x(t - t_d) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 (t - t_d)}$

$$= \sum_{k=-\infty}^{\infty} \left(a_k e^{-jk\omega_0 t_d}\right) e^{jk\omega_0 t}$$

This is the form of a Fourier series with coefficients

$$\boxed{b_k = a_k e^{-jk\omega_0 t_d}}$$

4.6 $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$; $a_0 = 0$ in this problem

$$y(t) = \int_0^t x(\tau)\,d\tau$$

(note τ, not t!)

(a)(i) $y(t) = \int_0^t \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 \tau}\right) d\tau$

$$= \sum_{k=-\infty}^{\infty} a_k \left(\int_0^t e^{jk\omega_0 \tau}\, d\tau\right)$$

$$= \sum_{k=-\infty}^{\infty} a_k \cdot \frac{1}{jk\omega_0} e^{jk\omega_0 \tau}\Big|_0^t$$

$$\boxed{y(t) = \sum_{k=-\infty}^{\infty} \frac{a_k}{jk\omega_0} e^{jk\omega_0 t} - \sum_{k=-\infty}^{\infty} \frac{a_k}{jk\omega_0}}$$

In general, the Fourier series for $y(t)$ is of the form

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 \tau}$$

$$= b_0 + \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} b_k e^{jk\omega_0 \tau}$$

(here I just pulled out the $k=0$ term)

Comparing this series to this series, we see that

$$b_0 = -\sum_{k=-\infty}^{\infty} \frac{a_k}{jk\omega_0}$$

(4.6 continued)

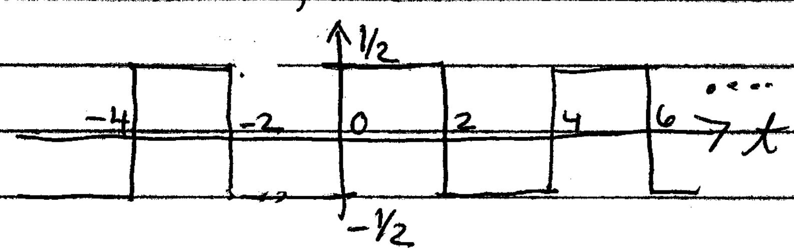
(a) (ii) for $k \neq 0$, $\boxed{b_k = \dfrac{a_k}{jk\omega_0}}$

(iii) If $a_0 \neq 0$, then $b_0 \to \infty$, which is bad, because the $k=0$ term in the formula for b_0 would be $\frac{a_0}{0}$.

With $a_0 \neq 0$, this term is indeterminate $\left(\frac{0}{0}\right)$. As stated in the problem, we'll consider ~~as~~ the term including a_0 to be zero.

(b) From the text, p. 52, with $a_0 = 0$ and $T_0 = 4$: $a_0 = 0$ means the integral over one period has to be zero, so half the square wave must be above the time axis, and half below:

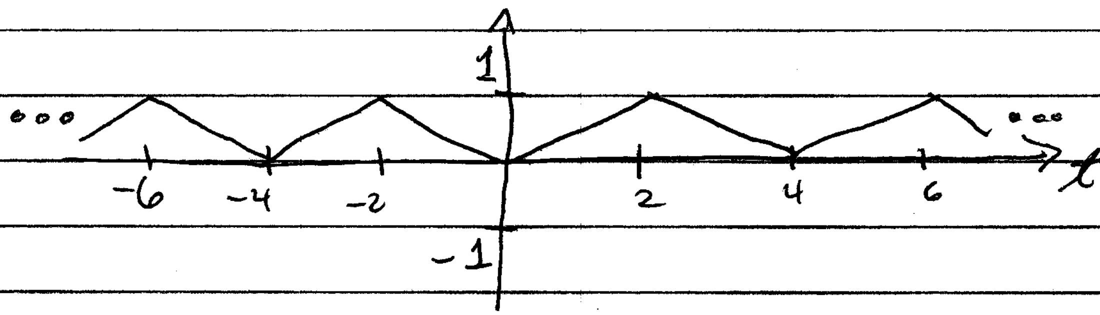
(i)



(ii) $y(t)$ is the integral from 0 to t of $x(\tau)$, which is the area ~~between~~ under the curve between 0 and t. What about when t is negative?

$\int_0^t f(\tau)\,d\tau = -\int_t^0 f(\tau)\,d\tau$ in general; that is,

we can integrate from the (negative) ~~t to zero and use the negative of that value.~~



(iii) We have $a_0 = 0$, so this matches conditions of part (a). ~~So we have~~ From p. 53, we know that

$$a_k = \begin{cases} \frac{1}{j\pi k}, & k = \pm 1, \pm 3, \pm 5, \dots \\ 0, & k = \pm 2, \pm 4, \pm 6, \dots \\ 0, & k = 0 \end{cases}$$

← this is the change that we made

So our "integration property" gives

$$\boxed{\begin{aligned} b_0 &= -\sum_{k=-\infty}^{\infty} \frac{a_k}{jk\omega_0} = +\sum_{\substack{k=-\infty \\ \cancel{k \text{ odd}} \\ k \neq 0}}^{\infty} \frac{1}{\pi\omega_0 k^2} \\ b_k &= \frac{a_k}{jk\omega_0} = \begin{cases} -\frac{1}{\pi\omega_0 k^2}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases} \end{aligned}}$$

$\omega_0 = \frac{2\pi}{T_0}$, but we have $T_0 = 4 \Rightarrow \omega_0 = \pi/2$

$$\Rightarrow \boxed{b_k = \begin{cases} \frac{-2}{\pi^2 k^2}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}}$$

Note that this matches Eqn. (3.39), p. 56 !!

Also, $b_0 = \sum_{\substack{k \text{ odd}, \\ k \neq 0}} \frac{2}{\pi^2 k^2} = \frac{2}{\pi^2} \sum_{\substack{k \text{ odd} \\ k \neq 0}} \frac{1}{k^2}$

$$= \frac{2}{\pi^2} \cdot 2 \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{1}{k^2}$$

$$= \frac{4}{\pi^2} \left\{ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots \right\}$$

You probably have to look this up in a table of series and integrals; you'll find that

$$\left\{ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots \right\} = \frac{\pi^2}{8}$$

So $\boxed{b_0 = \frac{4}{\pi^2}\left(\frac{\pi^2}{8}\right) = 1/2}$, which also agrees with Eqn. 3.39. Whew !!