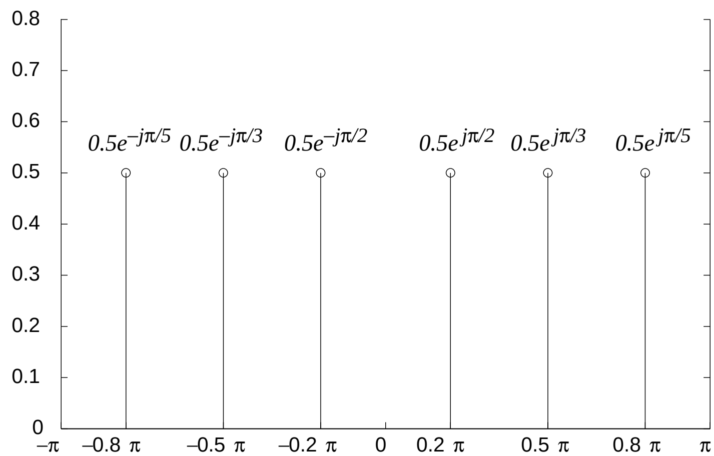
ECE2025
Fall 2003

Homework no. 5 Solutions

5.1

(a) $\hat{\omega} = \omega T_s = \omega / f_s$, so:

$$\begin{array}{lllll} \omega = 800\pi & \rightarrow & \hat{\omega} = 0.8\pi & & \\ \omega = 1500\pi & \rightarrow & \hat{\omega} = 1.5\pi & \rightarrow & -0.5\pi \quad \text{(folding)} \\ \omega = 2200\pi & \rightarrow & \hat{\omega} = 2.2\pi & \rightarrow & 0.2\pi \quad \text{(aliasing)} \end{array}$$



(b) $\omega = f_s \hat{\omega}$, so:

$$\begin{array}{lll} \hat{\omega} = 0.2\pi & \rightarrow & \omega = 200\pi \\ \hat{\omega} = 0.5\pi & \rightarrow & \omega = 500\pi \\ \hat{\omega} = 0.8\pi & \rightarrow & \omega = 800\pi \end{array}$$

$$y(t) = \cos(800\pi t + \pi/5) + \cos(500\pi t + \pi/3) + \cos(200\pi t + \pi/2)$$

(c) $\omega = 6000\pi \rightarrow \hat{\omega} = 6\pi \rightarrow 0$ and: $\omega = 12000\pi \rightarrow \hat{\omega} = 12\pi \rightarrow 0$, so: $y(t) = 4 - 3 - 1 = 0$.

(d) The angular frequencies of the sinusoids that make up $x(t)$ are 3500π and $3500\pi \pm 500\pi$. Therefore the largest angular frequency in $x(t)$ is $\omega_{max} = 4000\pi$, so the minimum sampling rate that will make $y(t) = x(t)$ is: $f_s = 2f_{max} = 2(\omega_{max}/2\pi) = 4000$ samples/sec.

5.2

(a) We must find values of f that, after sampling, get mapped either to 0.3π or to an alias of 0.3π or -0.3π.

1. First possibility: $\hat{\omega} = 0.3\pi$.
   $\hat{\omega} = 0.3\pi \rightarrow \omega = \hat{\omega} f_s = 4800\pi$ rad/sec ($f = 2400$ Hz); $\phi = \pi/4$.
   So: $x(t) = 3\cos(4800\pi t + \pi/4)$.
   Check:
   $$x(t) = 1.5e^{j\pi/4}e^{j4800\pi t} + 1.5e^{-j\pi/4}e^{-j4800\pi t}$$
   Sampling:
   $$\begin{aligned} e^{j4800\pi t} &\rightarrow e^{j(4800\pi/16000)n} = e^{j0.3\pi n} \\ 1.5e^{j\pi/4}e^{j4800\pi t} &\rightarrow 1.5e^{j\pi/4}e^{j0.3\pi n} \\ e^{-j4800\pi t} &\rightarrow e^{-j(4800\pi/16000)n} = e^{-j0.3\pi n} \\ 1.5e^{-j\pi/4}e^{-j4800\pi t} &\rightarrow 1.5e^{-j\pi/4}e^{-j0.3\pi n} \\ x(t) &\rightarrow 1.5e^{j\pi/4}e^{j0.3\pi n} + 1.5e^{-j\pi/4}e^{-j0.3\pi n} = 3\cos(0.3\pi n + \pi/4). \end{aligned}$$

2. Second possibility: $\hat{\omega} = -0.3\pi + 2\pi = 1.7\pi$ (an alias of -0.3π).
   $\hat{\omega} = 1.7\pi \rightarrow \omega = \hat{\omega} f_s = 27200\pi$ rad/sec ($f = 13600$ Hz); $\phi = -\pi/4$.
   (in this case sampling causes f to be aliased to a negative digital frequency, which means that this is a case of folding, so we must change the sign of ϕ). So: $x(t) = 3\cos(27200\pi t - \pi/4)$.
   Check:
   $$x(t) = 1.5e^{-j\pi/4}e^{j27200\pi t} + 1.5e^{j\pi/4}e^{-j27200\pi t}$$
   Sampling:
   $$\begin{aligned} e^{j27200\pi t} &\rightarrow e^{j(27200\pi/16000)n} = e^{j1.7\pi n} \rightarrow e^{-j0.3\pi n} \quad \text{(folding)} \\ 1.5e^{-j\pi/4}e^{j27200\pi t} &\rightarrow 1.5e^{-j\pi/4}e^{-j0.3\pi n} \\ e^{-j27200\pi t} &\rightarrow e^{-j(27200\pi/16000)n} = e^{-j1.7\pi n} \rightarrow e^{j0.3\pi n} \quad \text{(folding)} \\ 1.5e^{j\pi/4}e^{-j27200\pi t} &\rightarrow 1.5e^{j\pi/4}e^{j0.3\pi n} \\ x(t) &\rightarrow 1.5e^{-j\pi/4}e^{-j0.3\pi n} + 1.5e^{j\pi/4}e^{j0.3\pi n} = 3\cos(0.3\pi n + \pi/4). \end{aligned}$$

3. Third possibility: $\hat{\omega} = 0.3\pi + 2\pi = 2.3\pi$ (an alias of 0.3π).
   $\hat{\omega} = 2.3\pi \rightarrow \omega = \hat{\omega} f_s = 36800\pi$ rad/sec ($f = 18400$ Hz); $\phi = \pi/4$.
   (no folding in this case, because f gets aliased to a positive digital frequency, so the sign of ϕ remains the same). So: $x(t) = 3\cos(36800\pi t + \pi/4)$.
   Check:
   $$x(t) = 1.5e^{j\pi/4}e^{j36800\pi t} + 1.5e^{-j\pi/4}e^{-j36800\pi t}$$

Sampling:

$$\begin{array}{rcl} e^{j36800\pi t} & \rightarrow & e^{j(36800\pi/16000)n} = e^{j2.3\pi n} \rightarrow e^{j0.3\pi n} \quad \text{(aliasing)} \\ 1.5e^{j\pi/4}e^{j27200\pi t} & \rightarrow & 1.5e^{j\pi/4}e^{j0.3\pi n} \\ e^{-j36800\pi t} & \rightarrow & e^{-j(36800\pi/16000)n} = e^{-j2.3\pi n} \rightarrow e^{-j0.3\pi n} \quad \text{(aliasing)} \\ 1.5e^{-j\pi/4}e^{-j36800\pi t} & \rightarrow & 1.5e^{-j\pi/4}e^{-j0.3\pi n} \\ x(t) & \rightarrow & 1.5e^{j\pi/4}e^{j0.3\pi n} + 1.5e^{-j\pi/4}e^{-j0.3\pi n} = 3\cos(0.3\pi n + \pi/4). \end{array}$$

5.3

(a) The angular velocity of the disk is $\omega = 2\pi \cdot 720/60 = 24\pi$ rad/sec (1 revolution = 2π rad and 1 min = 60 sec). Therefore the position of the spot is: $p(t) = re^{j(24\pi t+\phi)}$.

(b) If the strobe is flashed every T_s seconds, the position of the spot at the n-th flash is: $p(nT_s) = re^{j(24\pi T_s n+\phi)}$. The spot will appear still if the disk makes an integer number of revolutions every T_s seconds, that is, if ωT_s is an integer multiple of 2π:

$$\omega T_s = 2k\pi, \qquad \text{or: } f_s = \frac{1}{T_s} = \frac{\omega}{2k\pi} = \frac{24\pi}{2k\pi} = \frac{12}{k}.$$

Since f_s is restricted to being an integer, k (and hence f_s) must be a divisor of 12. Therefore the possible values for f_s are: 1, 2, 3, 4, 6 and 12 samples/sec.

(c) If $f_s = 13$ samples/sec, then: $\hat{\omega} = \omega/f_s = 24\pi/13 > \pi$. Therefore this value gets aliased to: $\hat{\omega} = 24\pi/13 - 2\pi = -2\pi/13$, and the spot will appear to move at an angular frequency: $\omega = \hat{\omega} f_s = -2\pi$ rad/sec $= -1$ rps $= -60$ rpm (i.e. the spot will appear to be rotating clockwise instead of conterclockwise). The apparent position of the spot is given by:

$$p_a[n] = re^{j(-2\pi/13n+\phi)}.$$

It is easy to verify that this is the same as:

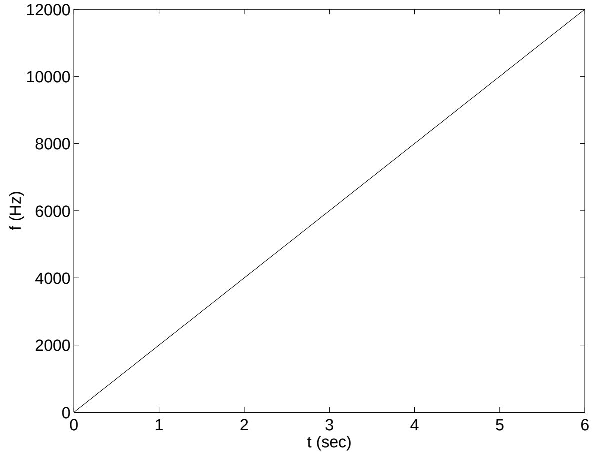
$$p(nT_s) = re^{j(24\pi/13n+\phi)}.$$

To see why, note the following sequence of equalities:

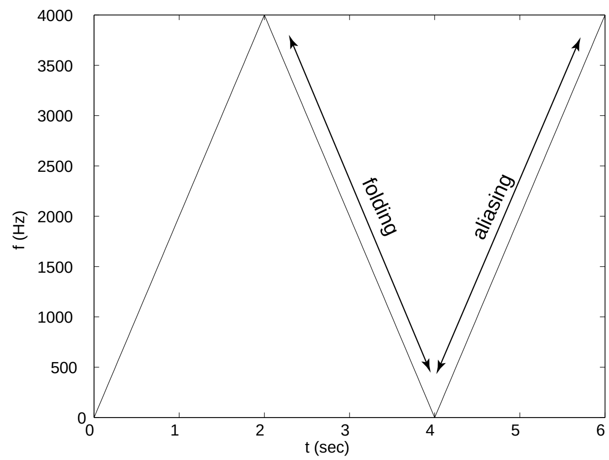
$$\begin{array}{rcl} p(nT_s) = p_a[n] & \Longleftrightarrow & re^{j(24\pi/13n+\phi)} = re^{j(-2\pi/13n+\phi)} \\ & \Longleftrightarrow & re^{j\phi}e^{j(24\pi/13)n} = re^{j\phi}e^{j(-2\pi/13)n} \\ & \Longleftrightarrow & e^{j(24\pi/13)n} = e^{j(-2\pi/13)n} \\ & \Longleftrightarrow & e^{j(26\pi/13)n} = 1 \\ & \Longleftrightarrow & e^{j2\pi n} = 1. \end{array}$$

5.5

(a) $\psi(t) = 2000\pi t^2$, so: $f(t) = \frac{1}{2\pi}\frac{d}{dt}\psi(t) = 2000t$.



(b) $x(t)$ is being sampled at a rate: $f_s = 8000$ samples/sec. Therefore aliasing (folding) will occur after the instantaneous frequency of the signal exceeds: $f_s/2 = 4$ KHz at $t = 2$ sec. After that, the frequency of the reconstructed signal (which is the frequency shown by `specgram`) will be: $|f(t) - f_s| = |2000t - 8000|$ (the absolute value is due to the fact that `specgram` only shows positive frequencies).



(c) The pitch of the sound that is heard increases until it reaches 4 KHz at $t = 2$ sec, then it starts decreasing until it reaches zero at $t = 4$ sec, and then it starts increasing again and it reaches 4 KHz at $t = 6$ sec.

5.6

(a) If $T_s = T = 1/f$, then:

$$\begin{aligned} x[n] &= x(nT) = \frac{1}{2}\left(e^{j(2\pi f T n+\phi)} + e^{-j(2\pi f T n+\phi)}\right) \\ &= \frac{1}{2}\left(e^{j\phi}e^{j2\pi n} + e^{-j\phi}e^{-j2\pi n}\right) = \frac{1}{2}\left(e^{j\phi} + e^{-j\phi}\right) = \cos\phi. \end{aligned}$$

Therefore: $y(t) = \cos\phi$.

(b) This is one possible approach to this problem. Sampling with $T_s = T/2 = 1/(2f)$:

$$\begin{aligned} x[n] &= \cos(2\pi f(T/2)n + \phi) = \cos(\pi n + \phi) \\ &= \cos\phi\cos\pi n - \sin\phi\sin\pi n \\ &= \cos\phi\cos\pi n \qquad (\text{because } \sin\pi n = 0 \text{ for all } n) \\ &= 0.5\cos\phi\, e^{j\pi n} + 0.5\cos\phi\, e^{-j\pi n} \end{aligned}$$

Therefore: $y(t) = 0.5\cos\phi\, e^{j\pi f_s t} + 0.5\cos\phi\, e^{-j\pi f_s t} = \cos\phi\cos(2\pi f t)$.