Problem Set 7 Solution - ECE2025 - Fall 2003

Problem 7.1

The general procedure for solving this problem is outlined in several examples in Chapter 6. Since the input signal is a sum of complex exponentials (i.e. sinusoids), we can resort to the following procedure:

1. Evaluate $H(e^{j\hat{\omega}})$ for the frequency of each complex exponential; and
2. Scale the amplitude and shift the phase of each of those components by the amplitude and phase of $H(e^{j\hat{\omega}})$ evaluated at the frequency of that component.

VERY IMPORTANT: This procedure ONLY WORKS FOR COMPLEX EXPONENTIAL INPUT SIGNALS. Do not try it on any other type of input signal.

The input signal is:

$$x[n] = 4 + 3\cos\left(\frac{\pi}{3}n - \frac{\pi}{2}\right) + 3\cos\left(\frac{7\pi}{8}n\right) \tag{1}$$

This signal is a sum of three complex exponentials (if this statement is confusing, remind yourself what a complex exponential is). The three frequency components of these three complex exponentials are $\hat{\omega} = 0$, $\hat{\omega} = \pi/3$, and $\hat{\omega} = 7\pi/8$. We need to plug these into $H(e^{j\hat{\omega}})$ and evaluate it.

What is $H(e^{j\hat{\omega}})$? Rather than work it out again here, the frequency response for a filter with these coefficients was already worked out in Example 6-1:

$$H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}}(2 + 2\cos\hat{\omega}) \tag{2}$$

If we evaluate $H(e^{j\hat{\omega}})$ at the specified component frequencies (plugging them into the above equations), they are (with magnitude and phase also separated out):

$\hat{\omega}$	$H(e^{j\hat{\omega}})$	$\lvert H(e^{j\hat{\omega}})\rvert$	$\angle H(e^{j\hat{\omega}})$
0	$4e^{-j0}$	4	0
$\frac{\pi}{3}$	$3e^{-j\frac{\pi}{3}}$	3	$-\frac{\pi}{3}$
$\frac{7\pi}{8}$	$0.1522e^{-j\frac{7\pi}{8}}$	0.1522	$-\frac{7\pi}{8}$

Now we take each of the components of $x[n]$ and scale (multiply) it by the magnitude of $H()$ at the appropriate frequency and shift (add to the phase) it by the phase shift of $H()$ at the appropriate frequency. This is shown below, with the scaled/shifted portioned identified in bold-face:

$$y[n] = \mathbf{4} \cdot 4 + \mathbf{3} \cdot 3\cos\left(\frac{\pi}{3}n - \frac{\pi}{2} - \frac{\pi}{\mathbf{3}}\right) + \mathbf{0.1522} \cdot \cos\left(\frac{7\pi}{8}n - \frac{\mathbf{7}\pi}{\mathbf{8}}\right) \tag{3}$$

This can be compacted to:

$$y[n] = 16 + 9\cos\left(\frac{\pi}{3}n - \frac{5\pi}{6}\right) + 0.4567\cos\left(\frac{7\pi}{8}n - \frac{7\pi}{8}\right) \tag{4}$$

Your book re-expresses the answer as:

$$y[n] = 16 + 9\cos\left(\frac{\pi}{3}(n-1) - \frac{\pi}{2}\right) + 0.4567\cos\left(\frac{7\pi}{8}(n-1)\right) \tag{5}$$

I'm not sure why they stop here, why not simplify it even further:

$$y[n] = 16 + 9\sin\left(\frac{\pi}{3}(n-1)\right) + 0.4567\cos\left(\frac{7\pi}{8}(n-1)\right) \tag{6}$$

Problem 7.2

PART a) The input is a sinusoid (complex exponential), so we can follow the procedure we did for homework problem 7.1. The input signal only has a single input frequency, so we evaluate $H(e^{j\hat{\omega}})$ at $\hat{\omega} = \pi/6$:

$$\begin{aligned} H(e^{j\frac{\pi}{6}}) &= B\sin\left(2\frac{\pi}{6}\right)e^{j(\frac{\pi}{2} - \frac{\pi}{6}C)} && (7) \\ &= B\frac{\sqrt{3}}{2}e^{j(\frac{\pi}{2} - \frac{\pi}{6}C)} && (8) \end{aligned}$$

When the filter $H()$ is applied to $x[n]$, as in problem 7.1 we need to take the input $x[n]$ and scale the amplitude and shift the phase as specified by $H()$. Thus:

$$\begin{aligned} y[n] &= \mathbf{B}\frac{\sqrt{\mathbf{3}}}{\mathbf{2}} \cdot \frac{1}{\sqrt{3}}\cos\left(\frac{\pi}{6}n + \frac{\pi}{4} - \frac{\pi}{\mathbf{6}}\mathbf{C} + \frac{\pi}{\mathbf{2}}\right) && (9) \\ &= \frac{B}{2}\cos\left(\frac{\pi}{6}n + \frac{3\pi}{4} - \frac{\pi}{6}C\right) && (10) \end{aligned}$$

In the assigned problem, we are given:

$$y[n] = \frac{1}{8}\cos\left(\frac{\pi}{6}n - \frac{\pi}{4}\right) \tag{11}$$

We can now solve for B and C by setting equations 10 and 11 equal to each other, and solving separately for the magnitude and phase components:

Magnitude:

$$
\begin{aligned}
B &= 2 \cdot \frac{1}{8} & (12)\\
&= \frac{1}{4} & (13)
\end{aligned}
$$

Phase:

$$
\begin{aligned}
C &= \frac{-\frac{3\pi}{4} - \frac{\pi}{4}}{-\frac{\pi}{6}} & (14)\\
&= 6 & (15)
\end{aligned}
$$

PART b) From our answer in part (a), the impulse response in the frequency domain is:

$$
\begin{aligned}
H(e^{j\hat{\omega}}) &= \frac{1}{4}\sin(2\hat{\omega})e^{-j(\frac{\pi}{2}-6\hat{\omega})} & (16)\\
&= \frac{j}{4}\sin(2\hat{\omega})e^{(-j6\hat{\omega})} & (17)\\
&= \frac{j}{4}\left(\frac{e^{j2\hat{\omega}} - e^{-j2\hat{\omega}}}{2j}\right)e^{-j\hat{\omega}6} & (18)\\
&= \frac{1}{8}(e^{j2\hat{\omega}} - e^{-j2\hat{\omega}})e^{-j6\hat{\omega}} & (19)\\
&= \frac{1}{8}(e^{-j4\hat{\omega}} - e^{-j8\hat{\omega}}) & (20)\\
& & (21)
\end{aligned}
$$

The above is the frequency response of two delay elements. The impulse response is:

$$
h[n] = \frac{1}{8}\delta[n-4] - \frac{1}{8}\delta[n-8] \qquad (22)
$$

Problem 7.3

PART a) The filter H_1 is a 5-point averaged. This filter is a low-pass filter, since higher frequency (faster-changing) components are reduced in amplitude by the averaging with nearby points. The filter H_2 is a first difference filter, similar to what you saw in lab # 6. It computes the difference of two consecutive points. This is a form of high-pass filter, since higher frequency signals, which change more per unit time than lower frequency signals, will have a larger amplitude when processed by this filter. This filter is a simple numerical estimate of the time derivative of the signal.

For more info on a running average filter like H_1, see section 6-7 of your text. For more information about first-difference filters like H_2, see section 6-5.2 of your text.

PART b) These two filters are much higher-order (45 points!), and the complement of each other (i.e. H_1=1-H_2 in the frequency domain). Because of the complementary nature of the frequency response of these filters, $H_1 + H_2 = 1$, and thus y should look the same as x.

Problem 7.4

PART a) The impulse response is:

$$h[n] = \sum_{k=-2}^{2} b_k \delta(n-k) \tag{23}$$

where $b[k]$ is:

k	-2	-1	0	1	2
b_k	-1	16	-30	16	-1

PART b) The filter is not causal, since it has coefficients for $n < 0$.

PART c & d) Eq. 25 (or 26) is the solution to part c, Eq. 28 is the solution to part d.

$$\begin{aligned}
H(e^{j\hat{\omega}}) &= \sum_{k=-2}^{2} h[k] e^{-j\hat{\omega}k} && (24)\\
&= -e^{-j\hat{\omega}(-2)} + 16e^{-j\hat{\omega}(-1)} - 30e^{-j\hat{\omega}(0)} + 16e^{-j\hat{\omega}(1)} - e^{-j\hat{\omega}(2)} && (25)\\
&= -30 + 16(e^{j\hat{\omega}} + e^{-j\hat{\omega}}) - (e^{j2\hat{\omega}} + e^{-j2\hat{\omega}}) && (26)\\
&= -30 + 32\frac{(e^{j\hat{\omega}} + e^{-j\hat{\omega}})}{2} - 2\frac{(e^{j2\hat{\omega}} + e^{-j2\hat{\omega}})}{2} && (27)\\
&= -30 + 32\cos(\hat{\omega}) - 2\cos(2\hat{\omega}) && (28)
\end{aligned}$$

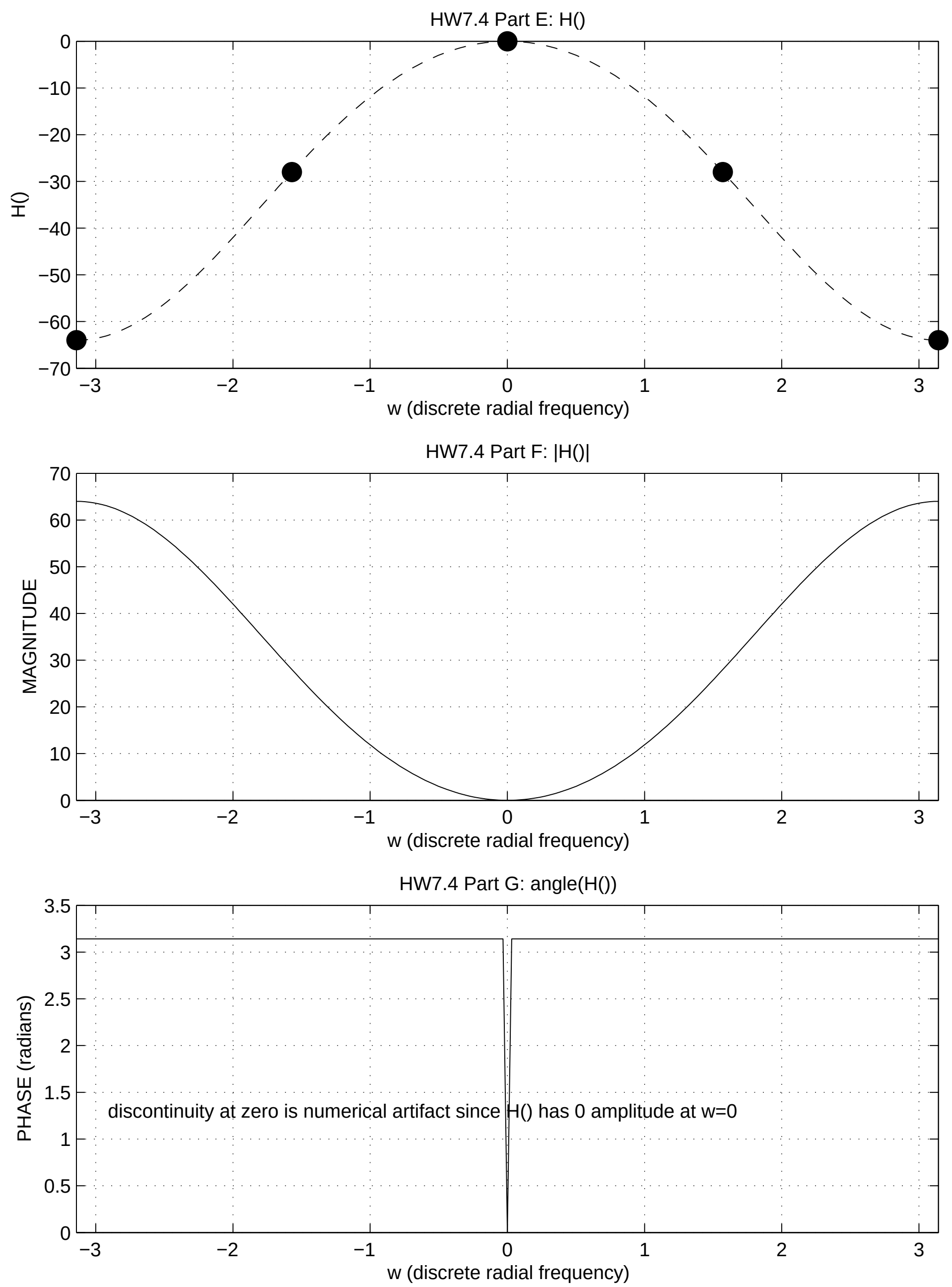
PART e) Did you notice that this frequency response only has real values? That is why we can plot it so easily. Tabulating different values of $H(e^{j\hat{\omega}})$ (the magnitude and phase are useful for parts f and g):

$\hat{\omega}$	$H(e^{j\hat{\omega}})$	$\lvert H(e^{j\hat{\omega}})\rvert$	$\angle H(e^{j\hat{\omega}})$
$-\pi$	-64	64	$-\pi$
$-\pi/2$	-28	28	$-\pi$
0	0	0	see note
$\pi/2$	-28	28	$-\pi$
π	-64	64	$-\pi$

Note: Something to ponder: Does phase mean anything if the magnitude is zero?

Parts e-g – see solution to 7.5.

Problem 7.5



HW7.4 Part E: H()
H()
w (discrete radial frequency)
HW7.4 Part F: |H()|
MAGNITUDE
w (discrete radial frequency)
HW7.4 Part G: angle(H())
PHASE (radians)
discontinuity at zero is numerical artifact since H() has 0 amplitude at w=0
w (discrete radial frequency)

Problem 7.6

PART a)

$$\begin{aligned} H(e^{j\hat{\omega}})H^*(e^{j\hat{\omega}}) &= (1-e^{j\hat{\omega}})\cdot(1-e^{-j\hat{\omega}}) && (29)\\ &= 1-e^{j\hat{\omega}}-e^{-j\hat{\omega}}+e^0 && (30)\\ &= 2-(e^{j\hat{\omega}}+e^{-j\hat{\omega}}) && (31)\\ &= 2-2\frac{(e^{j\hat{\omega}}+e^{-j\hat{\omega}})}{2} && (32)\\ &= 2(1-\cos(\hat{\omega})) && (33) \end{aligned}$$

PART b)

$$|H(e^{j\hat{\omega}})| = \sqrt{H(e^{j\hat{\omega}})H^*(e^{j\hat{\omega}})} = \sqrt{2(1-\cos(\hat{\omega}))} \qquad (34)$$

Problem 7.7

In your text, look at equation 6.25 on page 145, which is the frequency response of an L-point running average. We want to use this function, because the problem hint told us to, and because we know its impulse response (see text). Let's call this function $H_2(e^{j\hat{\omega}})$. It is of the form:

$$H_2(e^{j\hat{\omega}}) = \left(\frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}\right)e^{-j\hat{\omega}(L-1)/2} \qquad (35)$$

If we substitute in $L=8$, we find that:

$$H_2(e^{j\hat{\omega}}) = \left(\frac{\sin(4\hat{\omega})}{8\sin(\hat{\omega}/2)}\right)e^{-j\hat{\omega}3.5} \qquad (36)$$

This looks similar to the $H(e^{j\hat{\omega}})$ in your assignment, but not exactly. Since only the amplitude and exponential parts are different, let's assume that $H(e^{j\hat{\omega}}) = H_2(e^{j\hat{\omega}})H_3(e^{j\hat{\omega}})$. So we need to find a function $H_3(e^{j\hat{\omega}})$ that when multiplied by our function $H_2(e^{j\hat{\omega}})$ gives us $H(e^{j\hat{\omega}})$. In that case:

$$H_3(e^{j\hat{\omega}}) = \frac{H(e^{j\hat{\omega}})}{H_2(e^{j\hat{\omega}})} = \frac{16e^{-j6.5\hat{\omega}}}{\frac{1}{8}e^{-j3.5\hat{\omega}}} = 128e^{-j\hat{\omega}3} \qquad (37)$$

This is also a function that we know the impulse response of. So we can rewrite $H(e^{j\hat{\omega}})$ as:

$$H(e^{j\hat{\omega}}) = H_2(e^{j\hat{\omega}})H_3(e^{j\hat{\omega}}) = \left(\frac{\sin(4\hat{\omega})}{8\sin(\hat{\omega}/2)}e^{-j\hat{\omega}3.5}\right)\left(128e^{-j\hat{\omega}3}\right) \qquad (38)$$

Multiplication in the frequency domain is the same as convolution in the time domain, so $h[n] = h_2[n] * h_3[n]$. We know from p. 145 and what we've learned in class and recitation that $h_2[n]$ is an 8-pt averager and $h_3[n]$ is a unit delay with a gain of 128:

$$\begin{aligned} h_2[n] &= \frac{1}{8}\sum_{k=0}^{7}\delta[n-k] && (39)\\ h_3[n] &= 128\delta[n-3] && (40) \end{aligned}$$

The convolution of $h_2[n]$ and $h_3[n]$ is relatively straightforward, since $h_3[n]$ is just a 3-sample delay with a gain of 128. Therefore:

$$h[n] = h_2[n] * h_3[n] = 16 \sum_{k=3}^{10} \delta[n-k] \tag{41}$$

A plot of $h[n]$ is below.