ECE2025 Fall 2003
Solution of Problem Set #9

Problem 9.1:
Just read and sign.

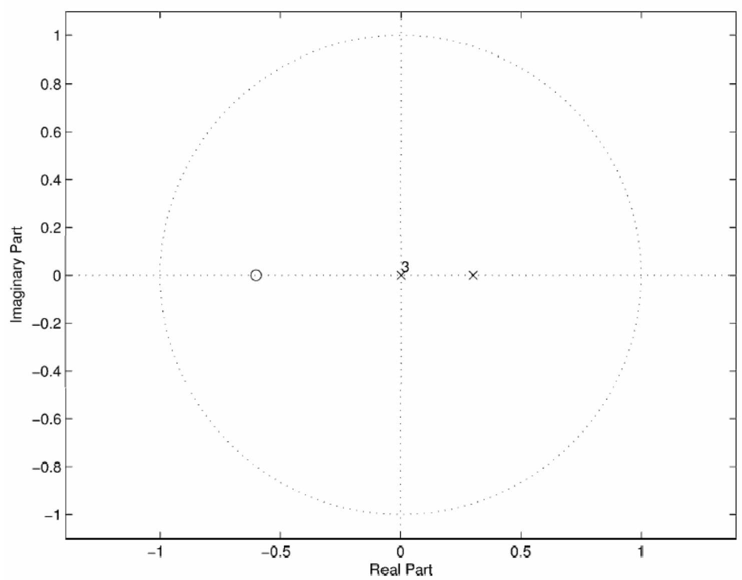
Problem 9.2:

(a) Taking z –transform to both sides of the input/output relationship,

$$Y(z) = 0.3z^{-1}Y(z) + z^{-3}X(z) + 0.6z^{-4}X(z),$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-3} + 0.6z^{-4}}{1 - 0.3z^{-1}} = \frac{z^{-3}(1 + 0.6z^{-1})}{1 - 0.3z^{-1}}$$

(b) The poles are $p_1 = 0$ (order of 3), $p_2 = 0.3$ and the zeros are $z_1 = -0.6$ and $z_2 = \infty$ (order of 3). The pole-zero plot is shown as the following.



(c) The z-transform of $x[n] = (-0.6)^n u[n]$ is

$$X(z) = \sum_{n=0}^{+\infty} (-0.6)^n z^{-n} = \frac{1}{1 + 0.6z^{-1}}.$$

Therefore, the z-transform of the system output will be

$$\begin{aligned} Y(z) &= H(z)X(z) \\ &= \frac{z^{-3}\left(1 + 0.6z^{-1}\right)}{1 - 0.3z^{-1}} \frac{1}{1 + 0.6z^{-1}} \\ &= \frac{z^{-3}}{1 - 0.3z^{-1}} \end{aligned}$$

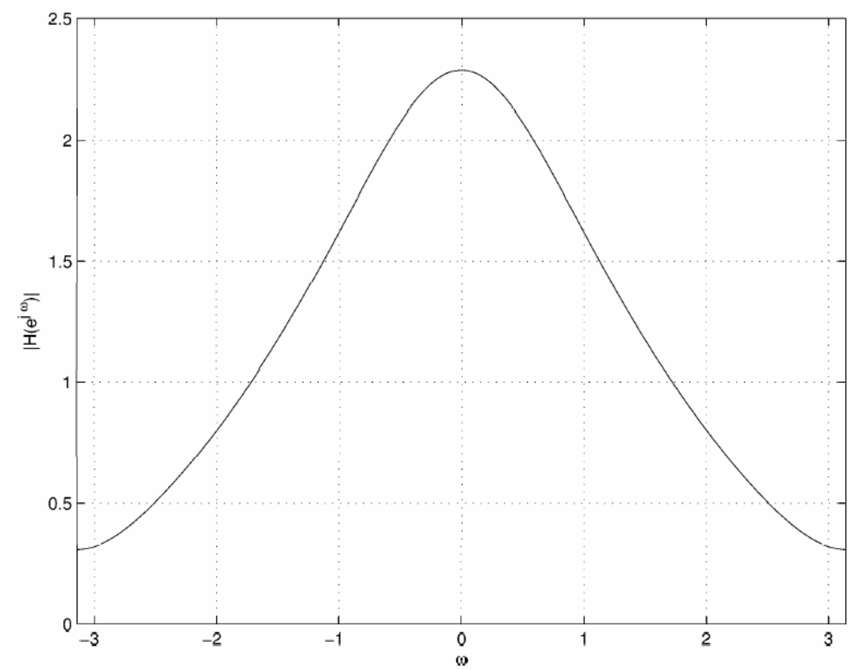
and

$$y[n]=(0.3)^{n-3}u[n-3].$$

(d) Setting $z=e^{j\hat{\omega}}$ in H(z), we have

$$H(z)=\frac{e^{-j3\hat{\omega}}\left(1+0.6e^{-j\hat{\omega}}\right)}{1-0.3e^{-j\hat{\omega}}}$$

(e) The plot of the magnitude is as the following:



Problem 9.3

(a) From H(z), we have

$$\frac{Y(z)}{X(z)}=\frac{1-0.2z^{-2}+0.3z^{-5}}{1+0.5z^{-3}-0.9z^{-7}}$$

$$\Updownarrow$$

$$\left(1+0.5z^{-3}-0.9z^{-7}\right)Y(z)=\left(1-0.2z^{-2}+0.3z^{-5}\right)X(z)$$

$$\Updownarrow$$

$$Y(z)=-0.5z^{-3}Y(z)+0.9z^{-7}Y(z)+X(z)-0.2z^{-2}X(z)+0.3z^{-5}X(z)$$

$$\Updownarrow$$

$$y[n]=-0.5y[n-3]+0.9y[n-7]+x[n]-0.2x[n-2]+0.3x[n-5]$$

(b) bb = [1, 0 - 0.2, 0, 0, 0.3] and aa = [1, 0 , 0, 0.5, 0, 0, 0, - 0.9].

Problem 9.4

(a) According to the *hint*, we first find inversion z-transform of

$$\tilde{H}(z) = \frac{1-\frac{\sqrt{3}}{4}z^{-1}}{1-\frac{\sqrt{3}}{2}z^{-1}+\frac{1}{4}z^{-2}}$$

$$= \frac{1-\frac{\sqrt{3}}{4}z^{-1}}{(1-\frac{1}{2}e^{j\frac{\pi}{6}}z^{-1})(1-\frac{1}{2}e^{-j\frac{\pi}{6}}z^{-1})}$$

$$= \frac{1/2}{1-\frac{1}{2}e^{j\frac{\pi}{6}}z^{-1}}+\frac{1/2}{1-\frac{1}{2}e^{-j\frac{\pi}{6}}z^{-1}},$$

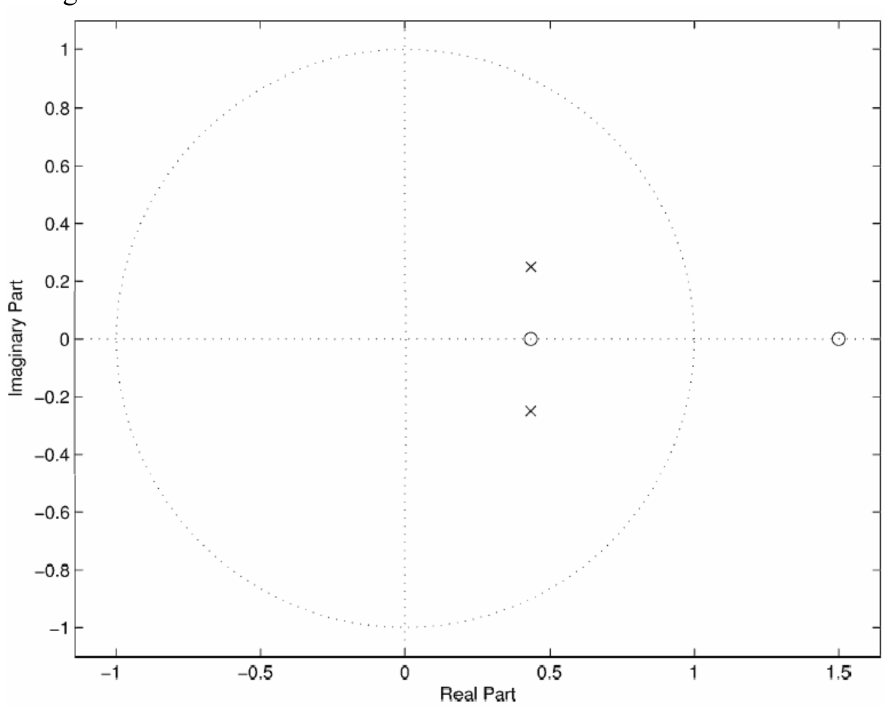
Therefore, the corresponding inverse z-transform of $\tilde{H}(z)$ will be

$$\tilde{h}[n] = \frac{1}{2}\left(\frac{1}{2}e^{j\frac{\pi}{6}}\right)^n u[n] + \frac{1}{2}\left(\frac{1}{2}e^{-j\frac{\pi}{6}}\right)^n u[n] = \left(\frac{1}{2}\right)^n \cos\left(\frac{\pi}{6}n\right)u[n].$$

Consequently, the inverse z-transform of $H(z) = (2-3z^{-1})\tilde{H}(z)$ will be

$$h[n] = 2\tilde{h}[n] - 3\tilde{h}[n-1] = 2\left(\frac{1}{2}\right)^n \cos\left(\frac{\pi}{6}n\right)u[n] - 3\left(\frac{1}{2}\right)^{n-1} \cos\left(\frac{\pi}{6}(n-1)\right)u[n-1].$$

(b) The poles are $p_{1,2} = \frac{1}{2}e^{\pm j\frac{\pi}{6}} = \frac{\sqrt{3}\pm j}{4}$ and the zeros are $z_1 = \frac{3}{2}$ and $z_2 = \frac{\sqrt{3}}{4}$. The pole-zero plot is as the following.

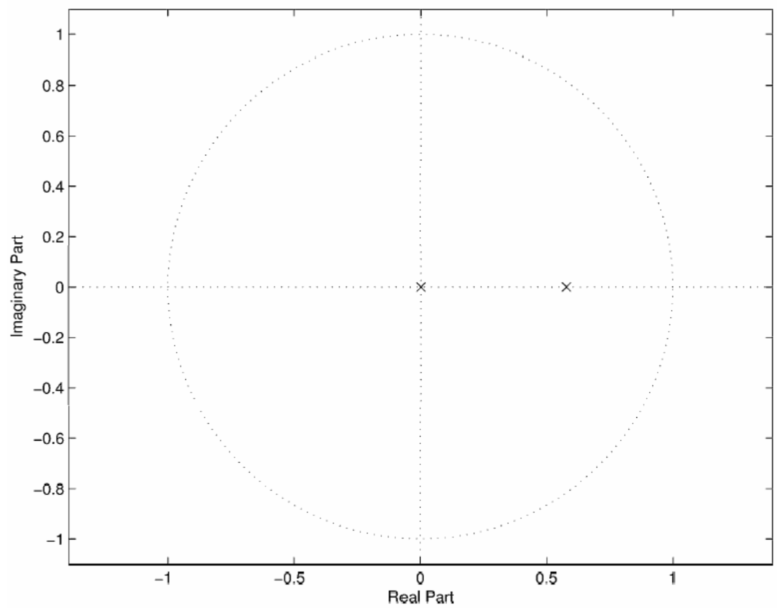


(c) All poles are inside the unit circle; hence the system is stable.

Problem 9.5.

(a) $h[n]=\left(\frac{1}{\sqrt{3}}\right)^{n-2}u[n-2]$.

(b) The poles are $p_1=0$ and $p_2=\frac{1}{\sqrt{3}}$ and the zero is $z=\infty$ (order of 2). The pole-zero plot is as the following.



(c)

$$H\left(e^{j\hat{\omega}}\right)=H(z)\big|_{z=e^{j\hat{\omega}}}=\frac{e^{-j2\hat{\omega}}}{1-\frac{1}{\sqrt{3}}e^{-j\hat{\omega}}}$$

(d)

$$x[n]=x(nT_s)=2\cos(0.5\pi n)$$

(e)

$$H\left(e^{j0.5\pi}\right)=\frac{e^{-j\pi}}{1-\frac{1}{\sqrt{3}}e^{-j0.5\pi}}=\frac{\sqrt{3}}{2}e^{-j\frac{7\pi}{6}}$$

$$\Downarrow$$

$$y[n]=2\left|H\left(e^{j0.5\pi}\right)\right|\cos(0.5\pi n+\angle H\left(e^{j0.5\pi}\right))=\sqrt{3}\cos(0.5\pi n-\frac{7\pi}{6})$$

(f)

$$y(t) = y[t/T_s] = \sqrt{3}\cos(500\pi t - \frac{7\pi}{6}).$$

(g)

$$x(t) = (\sqrt{3}-1)\cos(2000\pi t)$$
$$\Downarrow$$
$$x[n] = (\sqrt{3}-1)\cos(2000\pi n \bullet 0.001) = \sqrt{3}-1$$
$$\Downarrow$$
$$H(e^{j0}) = \frac{e^{-j0}}{1-\frac{1}{\sqrt{3}}e^{-j0}} = \frac{\sqrt{3}}{\sqrt{3}-1}$$
$$\Downarrow$$
$$y[n] = (\sqrt{3}-1)H(e^{j0}) = \sqrt{3}$$
$$\Downarrow$$
$$y(t) = \sqrt{3}$$

Problem 9.6

(a) It is a statement!

(b) From the input/output relation,

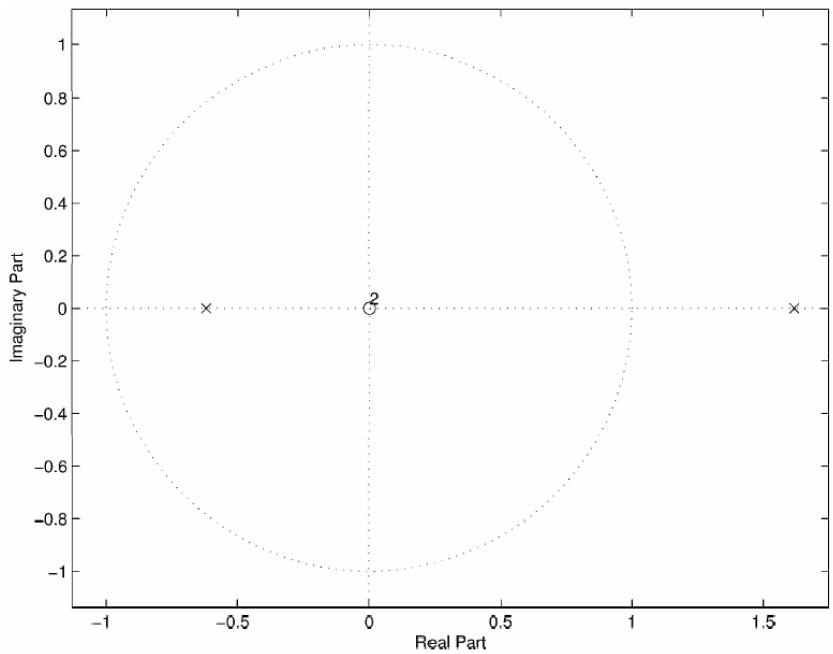
$$\begin{aligned} &y[0] = x[0] + y[-1] + y[-2] = \delta[0] + 0 + 0 = 1; \\ &y[1] = x[1] + y[0] + y[-1] = 0 + 1 + 0 = 1; \\ &y[2] = x[2] + y[1] + y[0] = 0 + 1 + 1 = 2; \\ &y[3] = x[3] + y[2] + y[1] = 0 + 2 + 1 = 3; \\ &\vdots \\ &y[3] = x[n] + y[n-1] + y[n-2] = y[n-1] + y[n-2];\ \text{for } n \geq 2, \end{aligned}$$

which confirms that y[n]'s are Fibonacci numbers.

(c) From the input/output relation, the system function will be

$$H(z) = \frac{1}{1-z^{-1}-z^{-2}}.$$

(d) The poles are $p_{1,2} = \frac{1 \pm \sqrt{5}}{2}$ and the zero is $z = 0$ (with order of 2). The pole-zero plot is as the following.



(e) From the input/output relation, the system output keeps on increasing; therefore, it is not stable. We can also see this from the poles of the system function in (d).

(f) It can be shown that

$$
\begin{aligned}
H(z) &= \frac{1}{1 - z^{-1} - z^{-2}} \\
&= \frac{1}{\left(1 - \frac{1+\sqrt{5}}{2} z^{-1}\right)\left(1 - \frac{1-\sqrt{5}}{2} z^{-1}\right)}, \\
&= \frac{\frac{1+\sqrt{5}}{2\sqrt{5}}}{1 - \frac{1+\sqrt{5}}{2} z^{-1}} - \frac{\frac{1-\sqrt{5}}{2\sqrt{5}}}{1 - \frac{1-\sqrt{5}}{2} z^{-1}}
\end{aligned}
$$

Therefore,

$$
h[n] = \frac{\sqrt{5}+1}{2\sqrt{5}} \left(\frac{\sqrt{5}+1}{2}\right)^n u[n] + \frac{\sqrt{5}-1}{2\sqrt{5}} \left(-\frac{\sqrt{5}-1}{2}\right)^n u[n],
$$

from which we can also have

$$
\begin{aligned}
y[0] &= 1; \\
y[1] &= 1; \\
y[2] &= 2; \\
y[3] &= 3.
\end{aligned}
$$