Solution to Problem Set #10

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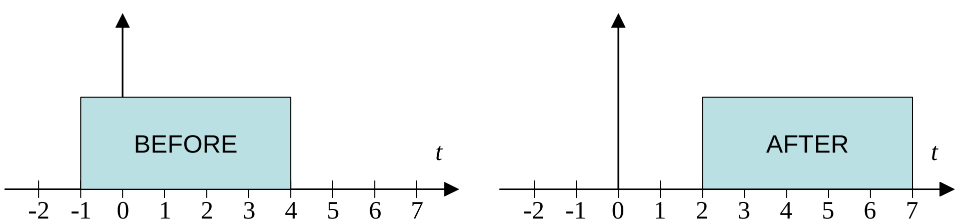
PROBLEM 10.1

(a) Convolution with an impulse returns the same function: $\delta(t) * x(t) = x(t)$;

Shift-invariance gives $\delta(t-t_0) * x(t) = x(t-t_0)$; so

$$\begin{aligned}\delta(t-3) * \{u(t+1) - u(t-4)\} &= \delta(t-3) * u(t+1) - \delta(t-3) * u(t-4) \\ &= u(t-3+1) - u(t-3-4) \\ &= u(t-2) - u(t-7)\end{aligned}$$

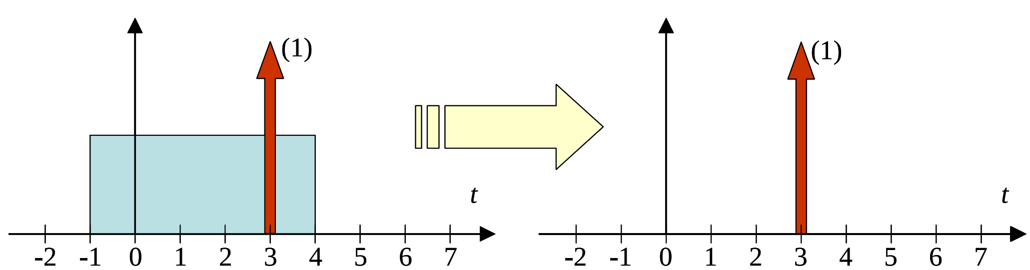


(b) Note that this is <u>multiplication</u> with an impulse, not convolution with an impulse, so the "sampling property" (Eqn. 9-16 in *SP First*) applies.

$$\delta(t-3)\{u(t+1) - u(t-4)\}$$
$$= u(3+1)\delta(t-3) - u(3-4)\delta(t-3) = u(4)\delta(t-3) - u(-1)\delta(t-3);$$

but $u(4) = 1$, and $u(-1) = 0$, so

$$\begin{aligned}\delta(t-3)\{u(t+1) - u(t-4)\} &= (1)\delta(t-3) - (0)\delta(t-3)\delta(t-3) \\ &= \delta(t-3)\end{aligned}$$



(c) From p. 261 of *SP First*, we have $\delta^{(1)}(t) * x(t) = x^{(1)}(t) = \frac{d}{dt}x(t)$, so

$$\delta^{(1)}(t) * \sin(100\pi t) = \frac{d}{dt}\sin(100\pi t) = 100\pi\cos(100\pi t)$$

(d) The hint refers to Eqn. (9.49) in *SP First*. Convolving an impulse with a delayed impulse just gets you the first impulse back again, delayed by the same amount, just as if you were convolving a delayed impulse with any normal function. Here we have three impulses convolved together, but we break it down into two successive convolutions:

$$\begin{aligned}\delta(t+7) * \delta(t-3) * \delta(t-11) &= \{\delta(t+7) * \delta(t-3)\} * \delta(t-11)\\ &= \delta(t+4) * \delta(t-11)\\ &= \delta(t-7)\end{aligned}$$

which in the end amounts to just adding up all of the shifts: $\delta(t+7-3-11) = \delta(t-7)$.

(e) Given $x(t) = \int_{-\infty}^{t} \delta(\tau+3)\,d\tau$, first note that the impulse occurs at τ = -3; so the integrand is considered to be zero everywhere except at τ = -3. So if we integrate from -∞ to *t*, but *t* < -3, then we have nothing in the integrand and the result is zero. If we let *t* ≥ -3, then the integrand includes the impulse and we get 1 (Eqn. (9.8) in *SP First*, the "area" of an impulse). So the result is

$$x(t) = \begin{Bmatrix} 0, & t < -3 \\ 1, & t \ge -3 \end{Bmatrix} = u(t+3)$$

This is an example of Eqn. (9.24) in *SP First*.

(f) $x(t) = \int_{-\infty}^{-2} t\delta(\tau+3)\,d\tau$. Note that the '*t*' can be pulled out of the integral over τ, and that the region of integration includes the impulse at τ = -3 (if it didn't, the answer would be *x*(*t*) ≡ 0). Because the "area" of an impulse is 1, we have

$$\begin{aligned}x(t) &= \int_{-\infty}^{-2} t\delta(\tau+3)\,d\tau = t\int_{-\infty}^{-2} \delta(\tau+3)\,d\tau\\ &= t(1) = t\end{aligned}$$

(g) This is the same as part (f), except now the limits of integration do *not* include the location of the impulse, so

$$\begin{aligned}x(t) &= \int_{-\infty}^{-4} t\delta(\tau+3)\,d\tau = t\int_{-\infty}^{-4} \delta(\tau+3)\,d\tau\\ &= 0\end{aligned}$$

(h) (will this problem never end?)

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{2}{\sqrt{3}}\cos(2t)u\left(t-\frac{\pi}{6}\right)\right\} \\
&=\frac{2}{\sqrt{3}}\cos(2t)\bullet\frac{d}{dt}\left\{u\left(t-\frac{\pi}{6}\right)\right\}+\frac{d}{dt}\left\{\frac{2}{\sqrt{3}}\cos(2t)\right\}u\left(t-\frac{\pi}{6}\right) \\
&=\frac{2}{\sqrt{3}}\cos(2t)\bullet\delta\left(t-\frac{\pi}{6}\right)+\frac{4}{\sqrt{3}}\sin(2t)u\left(t-\frac{\pi}{6}\right) \\
&=\frac{2}{\sqrt{3}}\cos\left(\frac{\pi}{3}\right)\bullet\delta\left(t-\frac{\pi}{6}\right)+\frac{4}{\sqrt{3}}\sin(2t)u\left(t-\frac{\pi}{6}\right) \\
&=\frac{1}{\sqrt{3}}\delta\left(t-\frac{\pi}{6}\right)-\frac{4}{\sqrt{3}}\sin(2t)u\left(t-\frac{\pi}{6}\right)
\end{aligned}$$

(This is similar to the sort of calculation on p. 253 and Example 9-3 in *SP First*.)

PROBLEM 10.2

(a) To consider linearity, suppose we have two inputs $x_1(t)$ and $x_2(t)$. They produce the outputs $y_1(t)=\{A+x_1(t)\}\cos\omega_c t$ and $y_2(t)=\{A+x_2(t)\}\cos\omega_c t$. Now consider the combined input $x(t) = x_1(t) + x_2(t)$. The output will be

$$\begin{aligned}
y(t)&=\{A+x(t)\}\cos\omega_c t \\
&=\{A+x_1(t)+x_2(t)\}\cos\omega_c t
\end{aligned}$$

The system is linear if $y(t)=y_1(t)+y_2(t)$ for any inputs $x_1(t)$ and $x_2(t)$ and for all t. (It's not adequate to be linear just for specific instances of time, or for specific well-chosen inputs.) This requires

$$\begin{aligned}
\{A+x_1(t)+x_2(t)\}\cos\omega_c t&=\{A+x_1(t)\}\cos\omega_c t+\{A+x_2(t)\}\cos\omega_c t \\
&=\{2A+x_1(t)+x_2(t)\}\cos\omega_c t
\end{aligned}$$

and this will only be true if $\boxed{\mathbf{A=0}}$. (Note that you should stop to think if there are any values of ω_c that will make this work, *e.g.* $\omega_c = 0$; in this case there aren't any such values.) If $A = 0$, then ω_c can be any value and it still works, so no constraint is needed on ω_c for linearity.

By the way, this system is a model of what goes on in AM radio modulation.

(b) To consider time invariance, define $x_0(t) = x(t-t_0)$. The corresponding output will be

$$
\begin{aligned}
y_0(t) &= \{A + x_0(t)\}\cos\omega_c t \\
&= \{A + x(t - t_0)\}\cos\omega_c t
\end{aligned}
$$

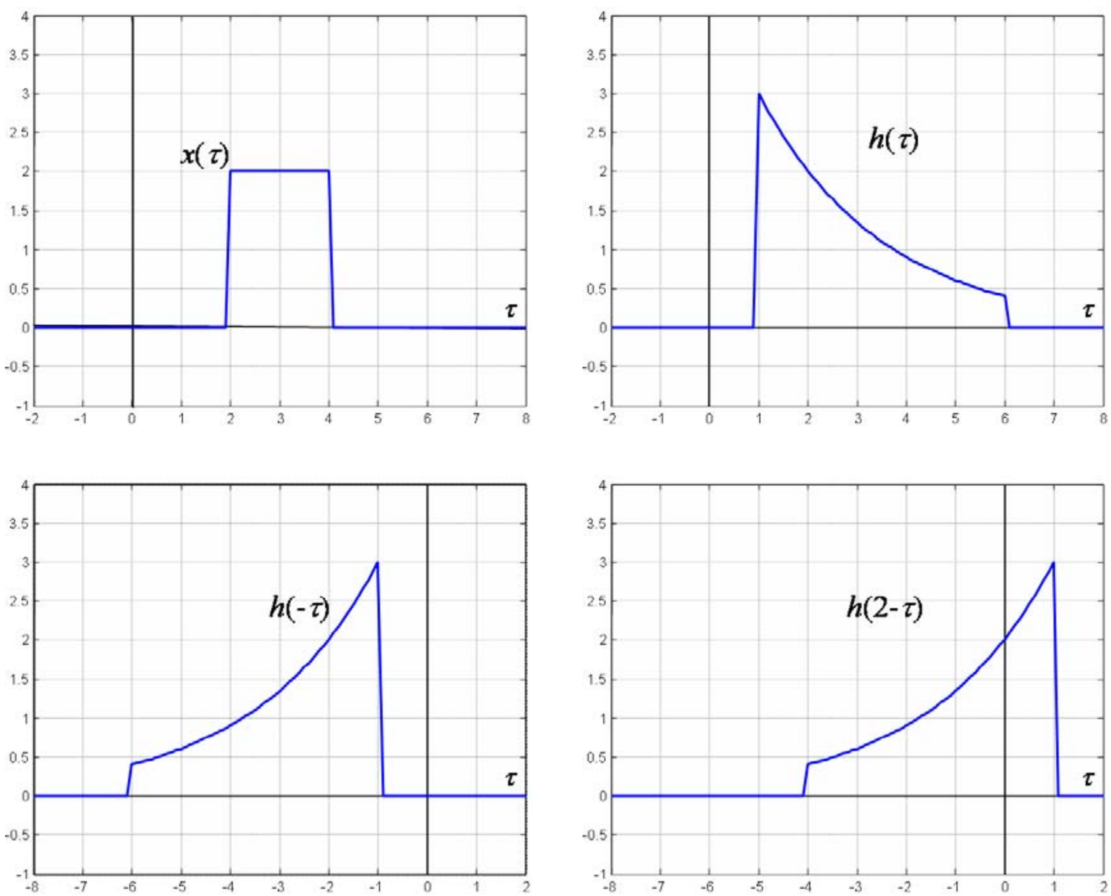
But

$$
\begin{aligned}
y(t - t_0) &= \{A + x(t - t_0)\}\cos\omega_c(t - t_0) \\
&= \{A + x(t - t_0)\}\cos(\omega_c t - \omega_c t_0)
\end{aligned}
$$

These two will only be the same, $y_0(t) = y(t - t_0)$ (again, for any inputs $x_1(t)$ and $x_2(t)$, for any t_0, and for all t) if $\cos(\omega_c t - \omega_c t_0) = \cos(\omega_c t)$ for all t. This appears to happen if $\omega_c t_0 = 2\pi k \Rightarrow \omega_c = 2\pi k / t_0$ for any integer k; but we would have to have a different ω_c for each t_0. The only value of ω_c that will work for *any* t_0 is $\boxed{\omega_c = 0}$. If this condition is satisfied, A can be any value.

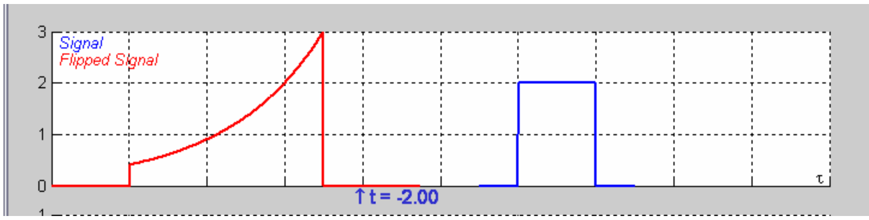
PROBLEM 10.3

(a)



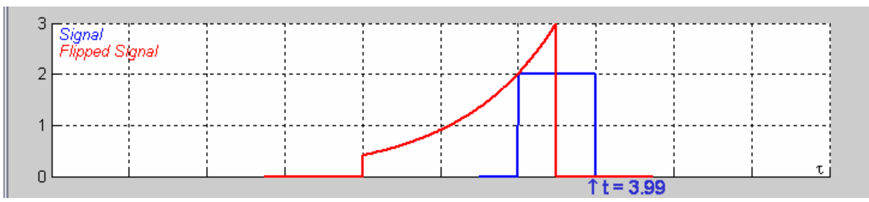
(b) Borrowing some pictures from `cconvdemo`:

Region 1: There is no overlap of $h(t-\tau)$ and $x(t)$ if t-1 < 2 → $\boxed{t<3}$:



$$y(t)=\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau=\int_{-\infty}^{+\infty}(0)\,d\tau=0$$

Region 2: Partial overlap if 2 < t-1 < 4 → $\boxed{3\le t<5}$:



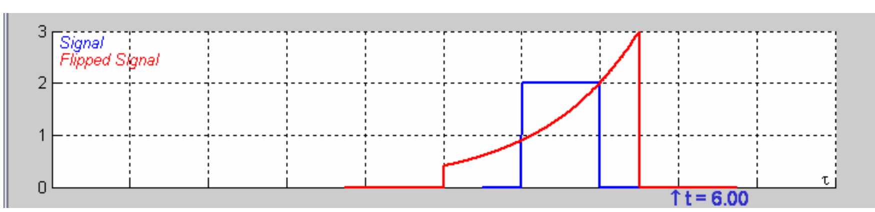
$$y(t)=\int_{2}^{t-1} x(\tau)h(t-\tau)\,d\tau=\int_{2}^{t-1}(2)\left(3e^{-0.4(t-\tau-1)}\right)d\tau=6e^{-0.4(t-1)}\int_{2}^{t-1}e^{+0.4\tau}\,d\tau$$

(Note that the integration limits cover only the range where the two functions overlap, because those are the only values of τ for which the integrand is not equal to zero. Also, I don't need the $u(t)$ part of $x(t)$ or $h(t)$ because my integration limits go only over the range where all of the step functions are nonzero and therefore equal to 1. Also, note how I changed 't' to '$t-\tau$' in the integrand, since I need $h(t-\tau)$, not just $h(t)$. Anyway, continuing:)

$$y(t)=6e^{-0.4(t-1)}\int_{2}^{t-1}e^{+0.4\tau}\,d\tau=\frac{6}{0.4}e^{-0.4(t-1)}\left\{e^{+0.4\tau}\right\}_{2}^{t-1}$$

$$=15e^{-0.4(t-1)}\left\{e^{+0.4(t-1)}-e^{+0.8}\right\}=15\left\{1-e^{-(0.4t-1.2)}\right\}$$

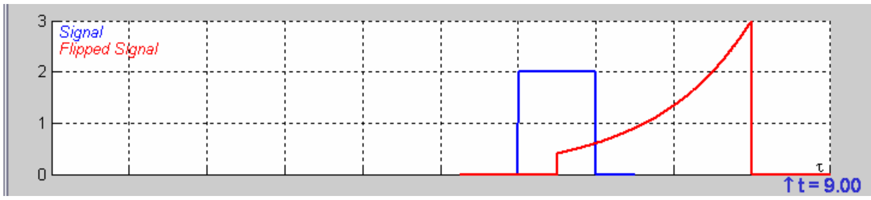
Region 3: Total overlap with t-1 > 4 and t-6 < 2→ $\boxed{5\le t<8}$:



Notice that the function in the integrand will be exactly the same every time; all that changes is the limits of integration. So we can pick it up at this point, but notice now that the limits have changed:

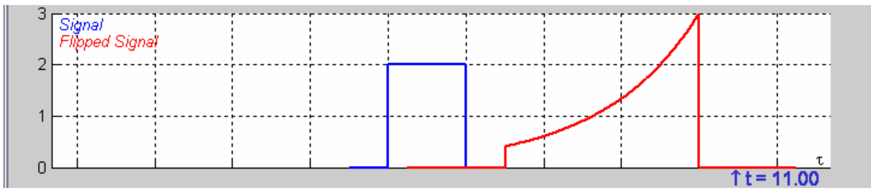
$$y(t)=15e^{-0.4(t-1)}\left\{e^{+0.4\tau}\right\}_2^4=15e^{-0.4t}\left\{e^{+2}-e^{+1.2}\right\}$$

Region 4: Partial overlap with t-6 > 2 and t-6 < 4→ $\boxed{8\le t<10}$:



$$y(t)=15e^{-0.4(t-1)}\left\{e^{+0.4\tau}\right\}_{t-6}^4=15\left\{e^{-(0.4t-2)}-e^{-2}\right\}$$

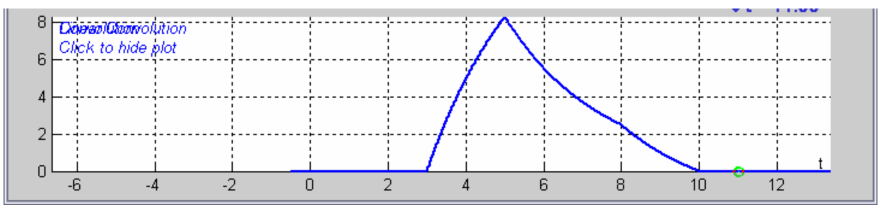
Region 5: No overlap with t-6 > 4 → $\boxed{t>10}$:



$$y(t)=0$$

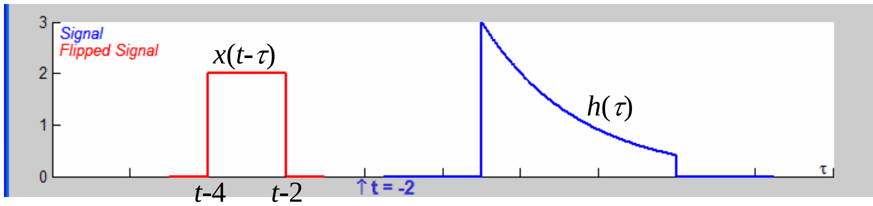
Putting it all together:

$$y(t)=\begin{cases}0 & t<3\\ 15\left\{1-e^{-(0.4t-1.2)}\right\} & 3\le t<5\\ 15e^{-0.4t}\left\{e^{+2}-e^{+1.2}\right\} & 5\le t<8\\ 15\left\{e^{-(0.4t-2)}-e^{-2}\right\} & 8\le t\le 10\\ 0 & t>10\end{cases}$$



PROBLEM 10.4

(a) The convolution integral is now $y(t)=\int_{-\infty}^{+\infty} h(\tau)x(t-\tau)\,d\tau$. A sketch of the integrand looks like this:



We can now step through the various regions just as we did in Problem 10.3(b). Here goes:

Region 1: No overlap with *t*-2 < 1→ $\boxed{t<3}$:

$$y(t)=0$$

Region 2: Partial overlap with *t*-2 > 1 and *t*-4 < 1→ $\boxed{3\le t<5}$:

$$y(t)=\int_{1}^{t-2} 3e^{-0.4(\tau-1)}(2)\,d\tau = 6e^{+0.4}\int_{1}^{t-2} e^{-0.4\tau}\,d\tau$$

$$=-15e^{+0.4}\int_{1}^{t-2} e^{-0.4\tau}\,d\tau = -15e^{+0.4}\left\{e^{-0.4\tau}\right\}_{1}^{t-2}$$

$$=-15e^{+0.4}\left(e^{-0.4(t-0.8)}-e^{-0.4}\right)=15\left(1-e^{-(0.4t-1.2)}\right)$$

Region 3: Partial overlap with *t*-2 < 6 and *t*-4 > 1→ $\boxed{5\le t<8}$:

$$y(t)=-15e^{+0.4}\left\{e^{-0.4\tau}\right\}_{t-2}^{t-4}=15e^{-0.4t}\left(e^{-2}-e^{-1.2}\right)$$

Region 4: Partial overlap with *t*-2 > 6 and *t*-4 < 6→ $\boxed{8\le t\le 10}$:

$$y(t)=-15e^{+0.4}\left\{e^{-0.4\tau}\right\}_{t-4}^{4}=15\left(e^{-0.4(t-2)}-e^{-2}\right)$$

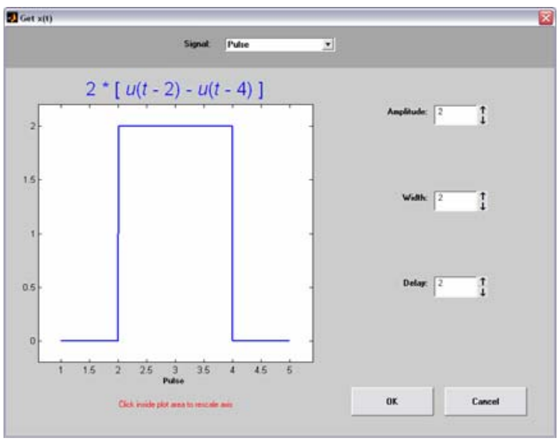
Region 5: No overlap with *t*-4 > 6 → $\boxed{t>10}$:

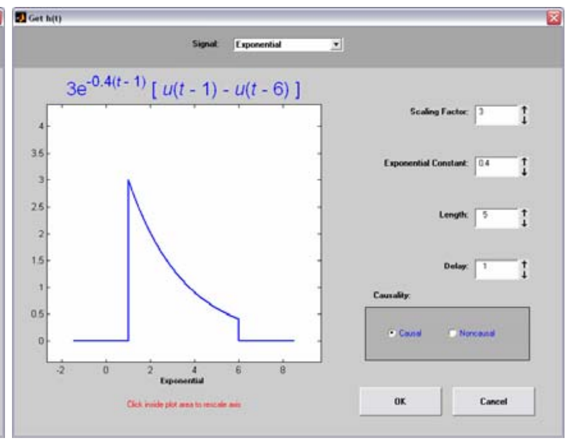
$$y(t)=0$$

Collecting the results, we see that they are identical to Problem 10.3(b):

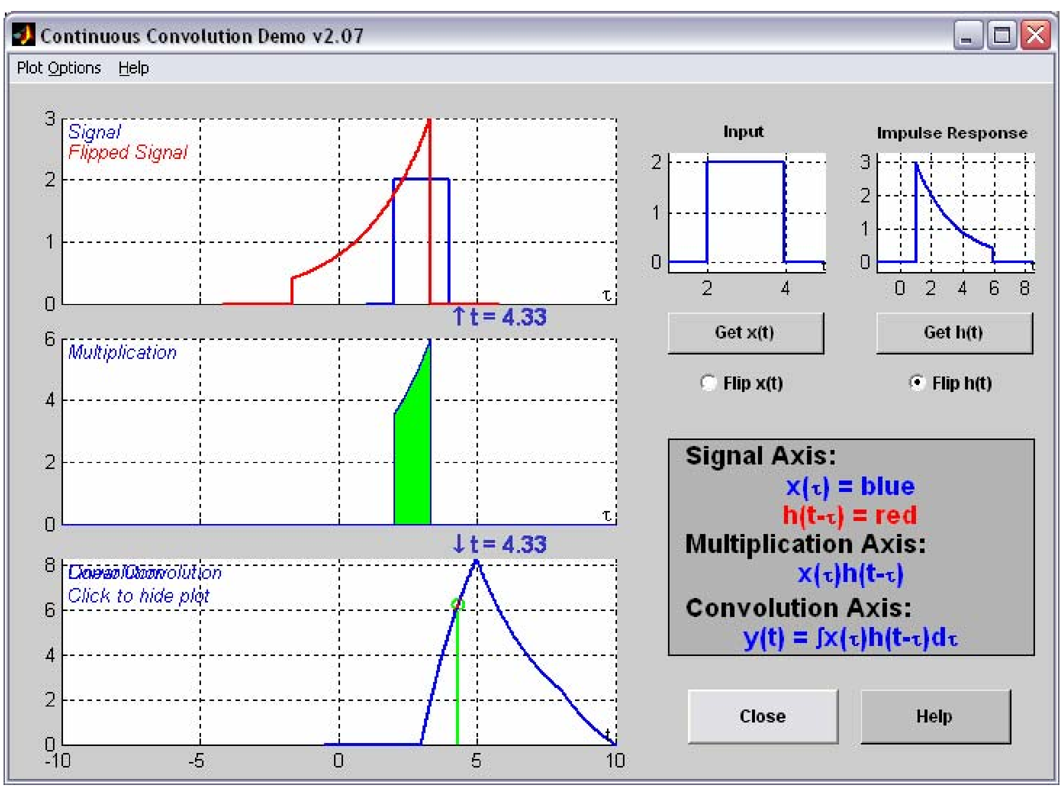
$$y(t)=\begin{cases}0 & t<3\\ 15\left\{1-e^{-(0.4t-1.2)}\right\} & 3\le t<5\\ 15e^{-0.4t}\left\{e^{-2}-e^{-1.2}\right\} & 5\le t<8\\ 15\left\{e^{-(0.4t-2)}-e^{-2}\right\} & 8\le t\le 10\\ 0 & t>10\end{cases}$$

(b) Here are some screen shots that show how to set up $x(t)$ and $h(t)$ in `cconvdemo`:



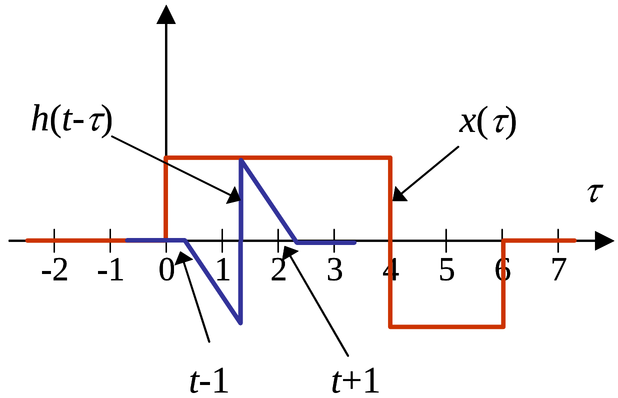


and this screen shot shows the calculation for t = 4.33:

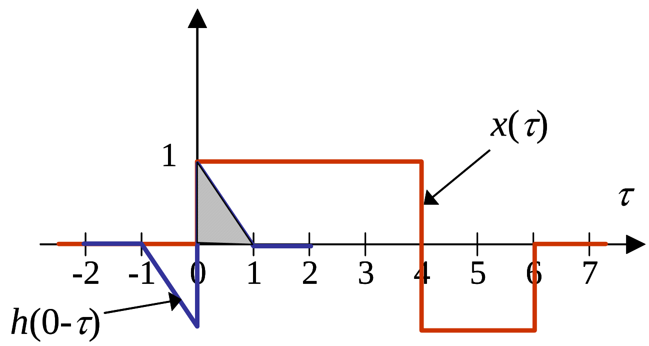


PROBLEM 10.5

$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)\,d\tau$ is the convolution integral. For the functions given, a typical drawing of the integrand looks like this (in this case, for *t* equal to about 1.2 or so):



(a) For computing *y*(0), we need the above figure for the specific case *t*=0. That looks like this:



The integral is the area of the product of *x*(τ) and *h*(0-τ), which is the shaded area in the plot immediately above. We can see by inspection that this area is ½, so

$$y(0) = \frac{1}{2}$$

(b) For a fixed value of *t*, *y*(*t*) is just the integral of the product of *x*(τ) and *h*(*t*-τ), that is, the *area* of the product. So *y*(*t*) = 0 when the area of the product is zero. That will happen whenever *x*(τ) and *h*(*t*-τ) don't overlap at all; this occurs if *t*+1 < 0 or if *t*-1 > 6.

We will also get a zero output when *h*(*t*-τ) is located at a place along the τ axis such that the nonzero part of *h*(*t*-τ) gets multiplied by a constant; due to the shape of *h*(*t*), the integral will then be zero. This happens whenever one of two things occurs:

- *h*(*t*-τ) is completely overlapping with the portion of *x*(τ) that has a value of +1. This will happen if *t*-1 ≥ 0 <u>and</u> *t*+1 ≤ 4 → $1 \le t \le 3$; or

- $h(t-\tau)$ completely overlaps the portion that has a value of -1. This will happen if t-1 ≥ 4 <u>and</u> t+1 ≤ 6 → t = 5 (a single point).

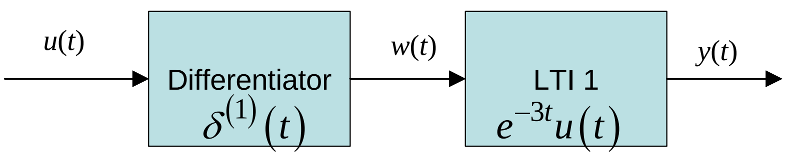
Summarizing: $y(t)=0$ when $\begin{cases} t<-1 \\ 1\le t\le 3 \\ t=5 \\ t>7 \end{cases}$

PROBLEM 10.6

(a)

$$
\begin{aligned}
h(t) &= h_1(t) * h_d(t) \\
&= \left\{e^{-3t}u(t)\right\} * \delta^{(1)}(t) = \frac{d}{dt}\left\{e^{-3t}u(t)\right\} \\
&= -3e^{-3t}u(t) + e^{-3t}\frac{d}{dt}u(t) = -3e^{-3t}u(t) + e^{-3t}\delta(t) \\
&= -3e^{-3t}u(t) + (1)\delta(t) = -3e^{-3t}u(t) + \delta(t)
\end{aligned}
$$

(b) Suppose we swap the order of the two subsystems; we can do that with LTI systems. This gives us this picture for this part of the problem:



Now we have the output of the first system, $w(t)$, is just an impulse:
$w(t)=\delta^{(1)}(t)*u(t)=\delta(t)$ (see Eqn. 9.22 in *SP First*); and since the input to the second system is now just an impulse, its output will be its impulse response, which is already given to us:

$$y(t)=e^{-3t}u(t)$$

(c) We already know the output in response to just $u(t)$ as an input from part (b), so by time invariance the output in response to an input of $u(t-10)$ is just
$y(t)=e^{-3(t-10)}u(t-10)$ and by linearity the response to $u(t)$ - $u(t-10)$ is

$$y(t)=e^{-3t}u(t)-e^{-3(t-10)}u(t-10)$$