

**EE-2025**

**Spring-2004**

**Lecture 6**

**Fourier Series Analysis**

**26-Jan-2004**

**General Info**

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- **Help Sessions: Su2, M6:15, T 6 and W6**
  - **Office Hours:** Visit any Prof or TA
  - **Bulletin Board: OFFICIAL ANNOUNCEMENTS**
- **Quiz #1 on 2-Feb**
  - Coverage: HW #1, #2, and #3
  - Old Quizzes & Problems are linked via WebCT
- Lab #3: **bring headphones**
  - Lots to read: Concept Maps
- Lab #2 report due at beginning of lab
- Prob Set #3 due This Week

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**The Rules**

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- Quizzes
  - NO make-ups given
  - Next Quiz would count for the one missed, IF excused
- Excused Absence
  - Must be written (by an “official”)
  - Notify ahead of time via e-mail
- Consult “INFO” on Web-CT for more details
  - Late Labs are –10 points per day
  - No late Homework

**Comments about Lab #2**

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- Direction Finding
  - Use **sensordatagen.p** to create test data
  - Compare angle from processing to true  $\theta$
- Source Localization
  - Beware of angles at  $\pi/2$ ,  $\tan \rightarrow$ infinity

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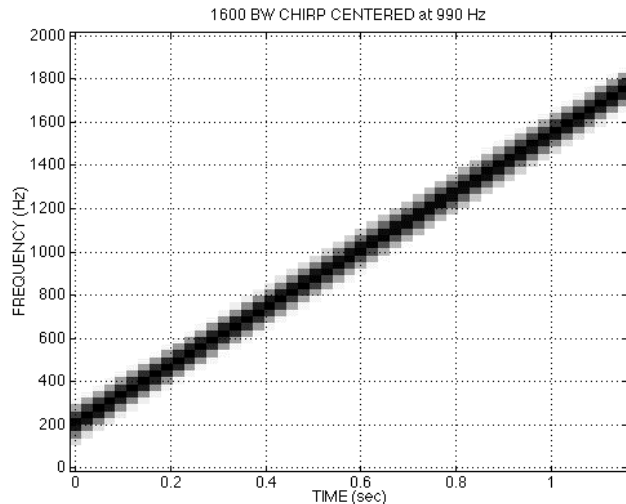
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# CHIRP SPECTROGRAM



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# Question: Create a Chirp

- Chirp should start at 200 Hz and end at 3200 Hz, and last for 1.5 sec.

$$x(t) = A \cos(\alpha t^2 + \beta t + \varphi)$$

$$\Rightarrow \alpha = ?$$

$$\Rightarrow \beta = ?$$

$$\Rightarrow \varphi = ?$$

$$\omega_i(t) = \frac{d}{dt} \psi(t) = 2\alpha t + \beta$$

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# READING ASSIGNMENTS

- This Lecture:
  - Fourier Series in Ch 3, Sects 3-4, 3-5 & 3-6**
    - Replaces pp. 62-66 in Ch 3 in DSP First
    - Notation:  $\mathbf{a}_k$  for Fourier Series
- Other Reading:
  - Next Lecture: More Fourier Series

# LECTURE OBJECTIVES

- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi k/T_0)t} dt$$

- ANALYSIS** via Fourier Series
  - For **PERIODIC** signals:  $\mathbf{x(t+T_0)} = \mathbf{x(t)}$
  - Later: spectrum from the Fourier Series

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# HISTORY

- Jean Baptiste Joseph Fourier
  - 1807 thesis (memoir)
    - On the Propagation of Heat in Solid Bodies
  - Heat !
  - Napoleonic era
- <http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html>



Joseph Fourier

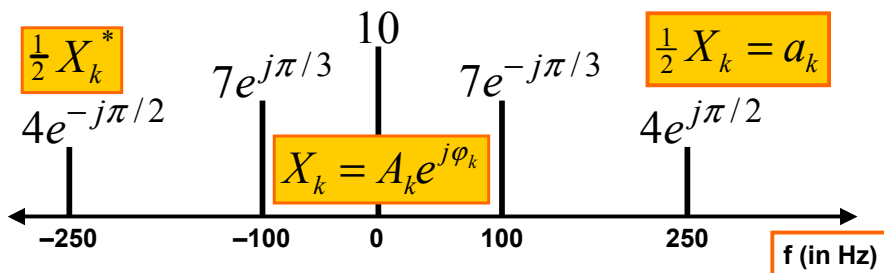
lived from 1768 to 1830

**Fourier** studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Find out more at:  
<http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fourier.html>

# SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = a_0 + \sum_{k=1}^N \{ a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \}$$

# Harmonic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

# Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

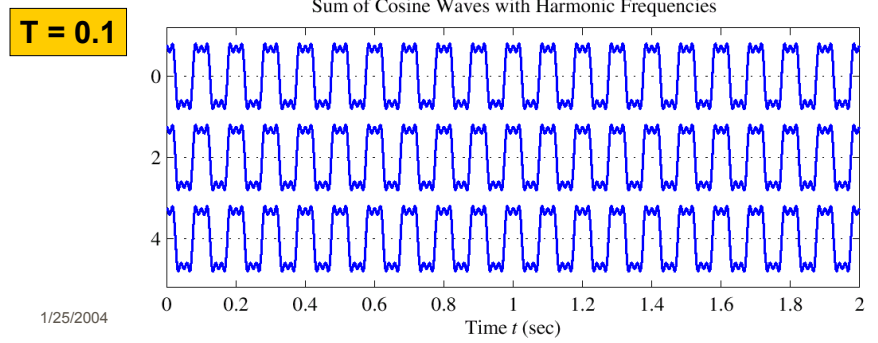
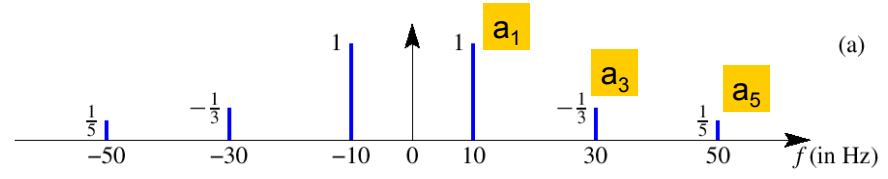
$$a_k = \frac{1}{2} X_k = \frac{1}{2} A_k e^{j\phi_k}$$

$$x(t) = A_0 + \sum_{k=1}^N A_k \cos(2\pi k f_0 t + \phi_k)$$

$$X_k = A_k e^{j\phi_k}$$

COMPLEX AMPLITUDE

# Harmonic Signal (3 Freqs)



# SYNTHESIS vs. ANALYSIS

- |   |   |
|---|---|
| <ul style="list-style-type: none"> <li>■ SYNTHESIS             <ul style="list-style-type: none"> <li>■ Easy</li> <li>■ Given <math>(\omega_k, A_k, \phi_k)</math> create <math>x(t)</math></li> </ul> </li> <li>■ Synthesis can be HARD             <ul style="list-style-type: none"> <li>■ Synthesize Speech so that it sounds good</li> </ul> </li> </ul> | <ul style="list-style-type: none"> <li>■ ANALYSIS             <ul style="list-style-type: none"> <li>■ Hard</li> <li>■ Given <math>x(t)</math>, extract <math>(\omega_k, A_k, \phi_k)</math></li> <li>■ How many?</li> <li>■ Need algorithm for computer</li> </ul> </li> </ul> |
|---|---|

# STRATEGY: $x(t) \rightarrow a_k$

- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

## INTEGRAL Property of exp(j)

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt} dt = \frac{T_0}{-j2\pi m} e^{-j(2\pi/T_0)mt} \Big|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m} (e^{-j2\pi m} - 1)$$

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt} dt = 0 \quad (m \neq 0) \quad \omega_0 = \frac{2\pi}{T_0}$$

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## ORTHOGONALITY of exp(j)

- PRODUCT of exp(+j ) and exp(-j )

$$\frac{1}{T_0} \int_0^{T_0} e^{j(2\pi/T_0)\ell t} e^{-j(2\pi/T_0)kt} dt = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$$

$$\frac{1}{T_0} \int_0^{T_0} e^{j(2\pi/T_0)(\ell-k)t} dt$$

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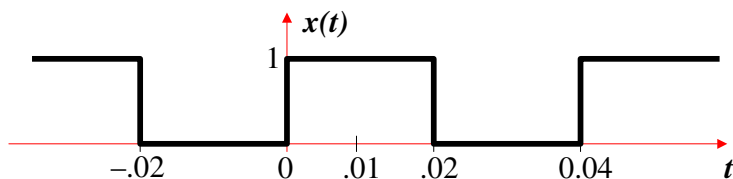
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## SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2}T_0 \\ 0 & \frac{1}{2}T_0 \leq t < T_0 \end{cases}$$

for  $T_0 = 0.04$  sec.



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## FS for a SQUARE WAVE {a<sub>k</sub>}

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k \neq 0)$$

$$a_k = \frac{1}{.04} \int_0^{.02} 1 e^{-j(2\pi/.04)kt} dt = \frac{1}{.04(-j2\pi k/.04)} e^{-j(2\pi/.04)kt} \Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)} (e^{-j(\pi)k} - 1) = \frac{1 - (-1)^k}{j2\pi k}$$

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## DC Coefficient: $a_0$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k = 0)$$

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt = \frac{1}{T_0} (\text{Area})$$

$$a_0 = \frac{1}{.04} \int_0^{.02} 1 dt = \frac{1}{.04} (.02 - 0) = \frac{1}{2}$$

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## Fourier Coefficients $a_k$

- $a_k$  is a function of  $k$ 
  - Complex Amplitude for  $k$ -th Harmonic
  - This one doesn't depend on the period,  $T_0$

$$a_k = \frac{1 - (-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$

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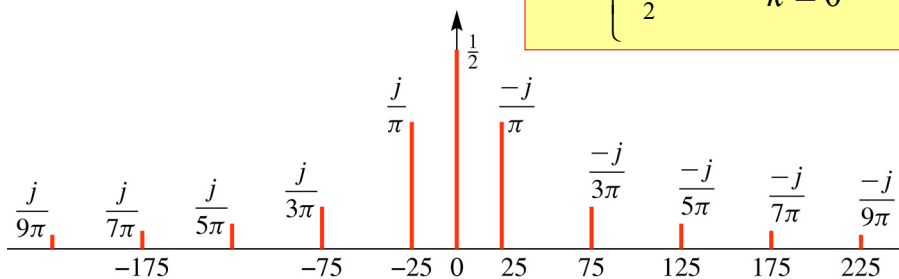
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## Spectrum from Fourier Series

$$\omega_0 = 2\pi / (0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \frac{-j}{\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$



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## Fourier Series Integral

- HOW do you determine  $a_k$  from  $x(t)$  ?

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

Fundamental Frequency  $f_0 = 1/T_0$

$a_{-k} = a_k^*$  when  $x(t)$  is real

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt \quad (\text{DC component})$$

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