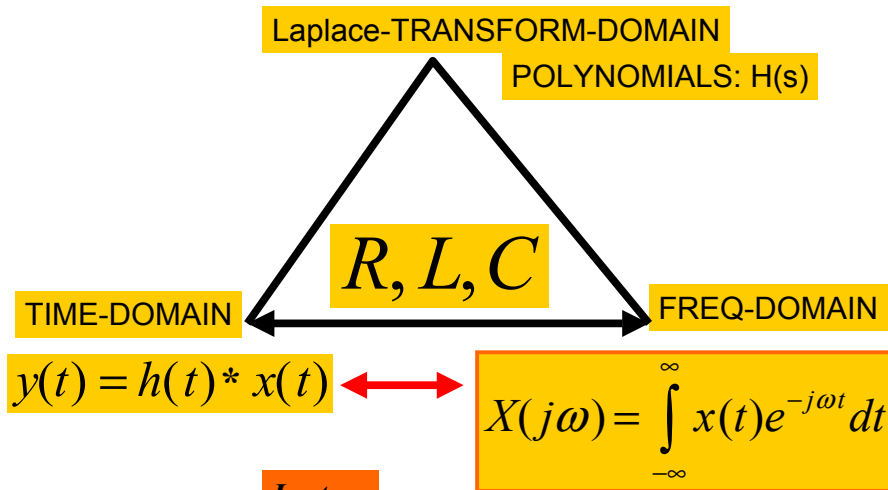


Lecture 19
Introduction to the Fourier Transform
26-Mar-04

Info: Web-CT, Lab, HW

- CHECK YOUR GRADES !!!
 - Web-CT is the OFFICIAL gradebook
- Quiz #3 will be 9-April (Friday)
 - Coverage: HW #8, 9, and 10
 - Chapters 7, 9, 10, and 11
 - Review Session, 8-April, Thurs @ 7:30pm

THREE DOMAINS: ANALOG



READING ASSIGNMENTS

- This Lecture:
 - Chapter 11, Sects. 11-1 to 11-4
- Other Reading:
 - Recitation: Ch. 10
 - And Chapter 11, Sects. 11-1 to 11-4
 - Next Lecture: Chapter 11, Sects. 11-5, 11-6

LECTURE OBJECTIVES

- Review
 - Frequency Response
 - Fourier Series
- Definition of **Fourier transform**

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

Relation to Fourier Series

- Examples of Fourier transform pairs

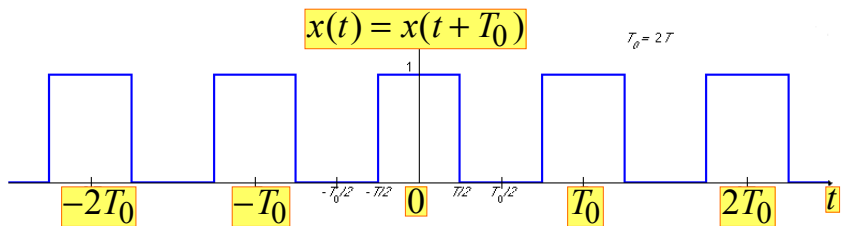
Everything = Sum of Sinusoids

- One Square Pulse = Sum of Sinusoids
 - ????????????
- Finite Length
- Not Periodic



- Limit of Square Wave as Period → infinity
 - Intuitive Argument

Fourier Series: Periodic $x(t)$



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

Fourier Synthesis

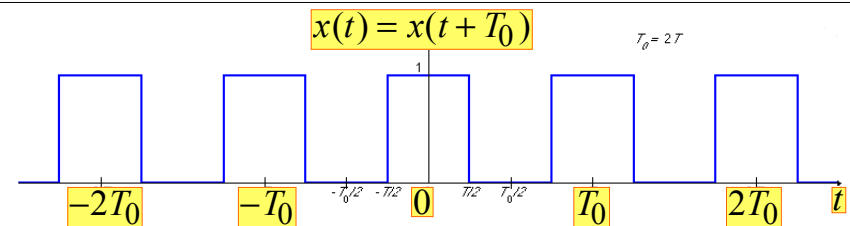
Fundamental Freq.

$$\omega_0 = 2\pi / T_0 = 2\pi f_0$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)e^{-j\omega_0 k t} dt$$

Fourier Analysis

Square Wave Signal



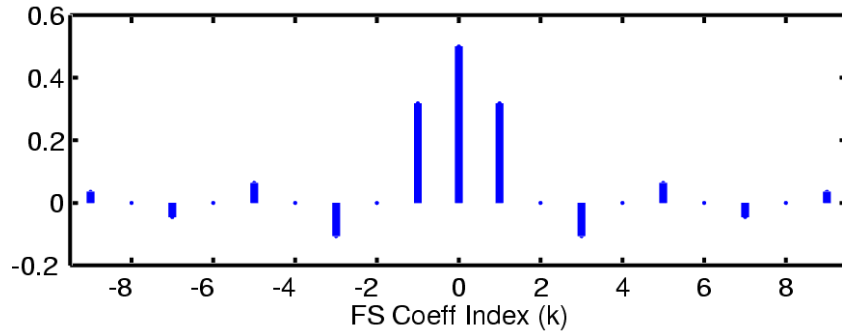
$$a_k = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} (1)e^{-j\omega_0 k t} dt$$

$$a_k = \frac{e^{-j\omega_0 k t} \Big|_{-T_0/4}^{T_0/4}}{-j\omega_0 k T_0} = \frac{e^{-j\pi k / 2} - e^{j\pi k / 2}}{-j2\pi k} = \frac{\sin(\pi k / 2)}{\pi k}$$

Spectrum from Fourier Series

$$a_k = \frac{\sin(\pi k / 2)}{\pi k} = \begin{cases} \neq 0 & k = 0, \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \end{cases}$$

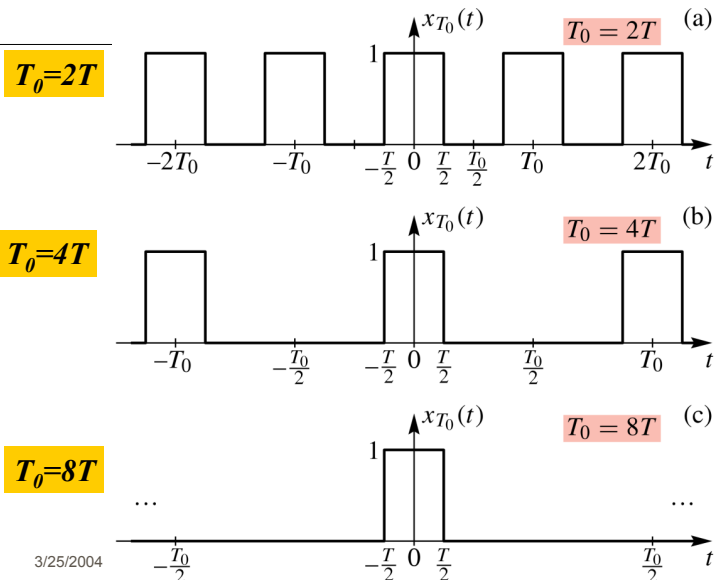
Fourier Series Coeffs for Square Wave



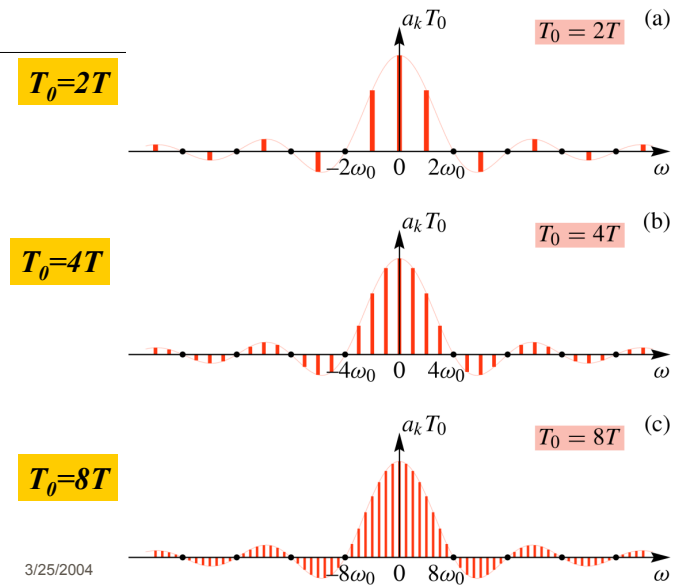
What if $x(t)$ is not periodic?

- Sum of Sinusoids?
 - Non-harmonically related sinusoids
 - Would not be periodic, but would probably be non-zero for all t .
- Fourier transform
 - gives a “sum” (actually an **integral**) that involves **ALL** frequencies
 - can represent signals that are identically zero for negative t . !!!!!!!!

Limiting Behavior of FS



Limiting Behavior of Spectrum



Plot
($T_0 a_k$)

FS in the LIMIT (long period)

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} (T_0 a_k) e^{j\omega_k t} \left(\frac{2\pi}{T_0}\right) \mapsto x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

$$\lim_{T_0 \rightarrow \infty} \frac{2\pi}{T_0} = d\omega$$

$$\lim_{T_0 \rightarrow \infty} \frac{2\pi}{T_0} k = \omega$$

$$\lim_{T_0 \rightarrow \infty} T_0 a_k = X(j\omega)$$

$$T_0 a_k = \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_k t} dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

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Fourier Transform Defined

- For non-periodic signals

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

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Example 1:

$$x(t) = e^{-at} u(t)$$

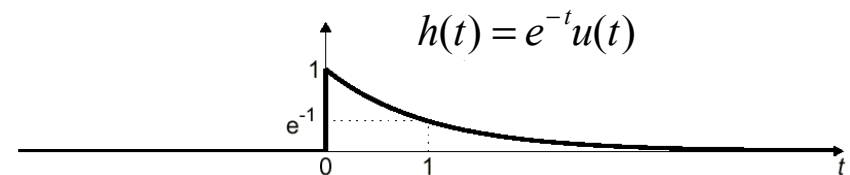
$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$X(j\omega) = -\frac{e^{-at} e^{-j\omega t}}{a+j\omega} \Big|_0^{\infty} = \frac{1}{a+j\omega} \quad a > 0$$

$$X(j\omega) = \frac{1}{a+j\omega}$$

Frequency Response

- Fourier Transform of $h(t)$ is the Frequency Response



$$h(t) = e^{-t} u(t) \Leftrightarrow H(j\omega) = \frac{1}{1+j\omega}$$

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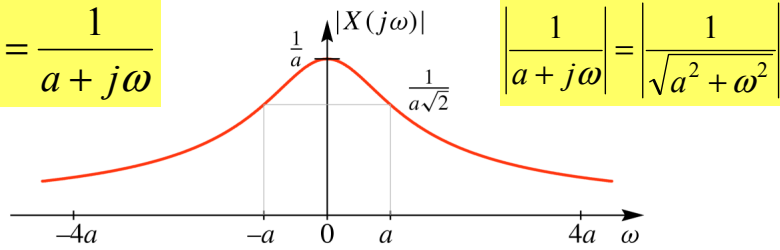
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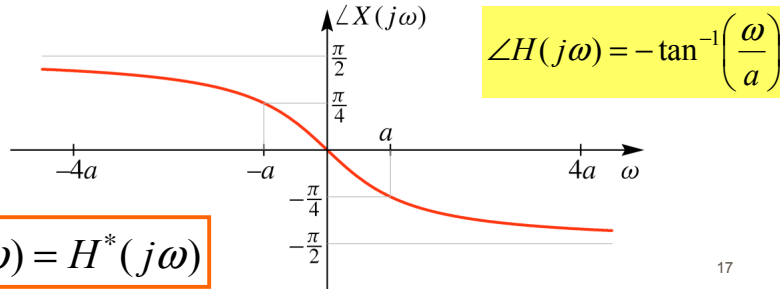
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Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{a + j\omega}$$



$$\left| \frac{1}{a + j\omega} \right| = \frac{1}{\sqrt{a^2 + \omega^2}}$$



$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

$$H(-j\omega) = H^*(j\omega)$$

Example 2:

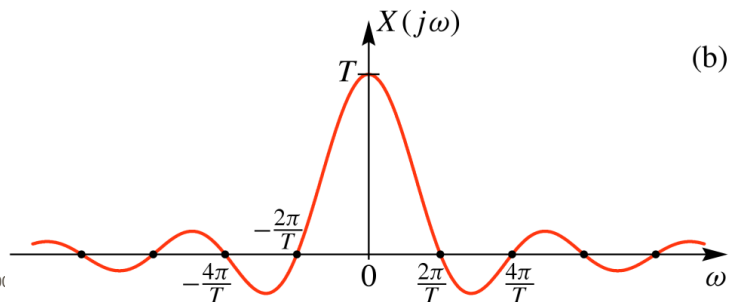
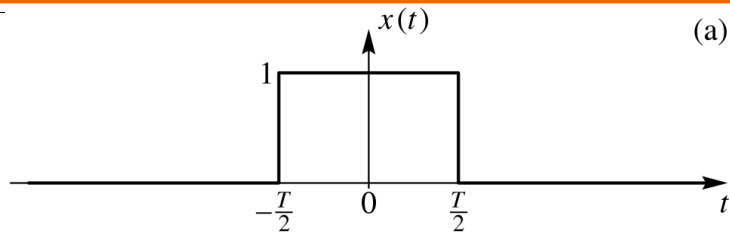
$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}$$

$$X(j\omega) = \int_{-T/2}^{T/2} (1)e^{-j\omega t} dt = \int_{-T/2}^{T/2} e^{-j\omega t} dt$$

$$X(j\omega) = \frac{e^{-j\omega t}}{-j\omega} \Big|_{-T/2}^{T/2} = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$X(j\omega) = \frac{\sin(\omega T / 2)}{(\omega / 2)}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T / 2)}{(\omega / 2)}$$



Example 3:

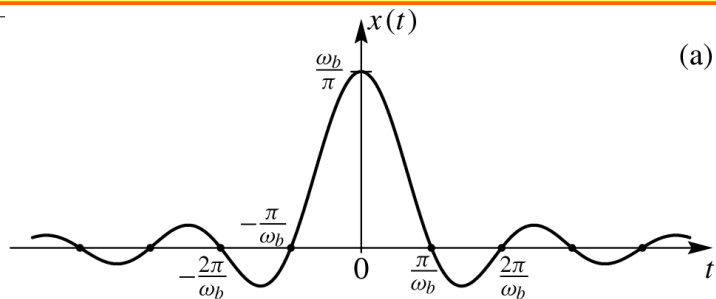
$$X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega t} d\omega$$

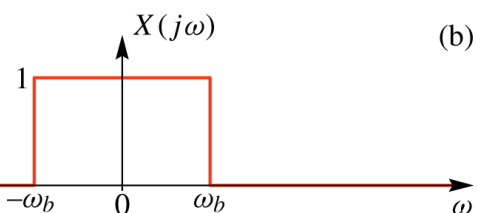
$$x(t) = \frac{1}{2\pi} \frac{e^{j\omega t}}{jt} \Big|_{-\omega_0}^{\omega_0} = \frac{1}{2\pi} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{jt}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$



(a)



(b)

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Example 4:

$$x(t) = \delta(t - t_0)$$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Shifting Property of the Impulse

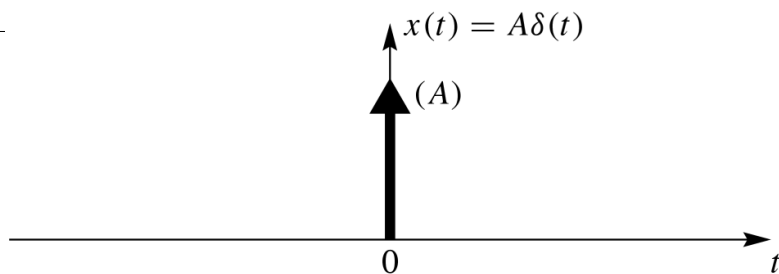
$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0}$$

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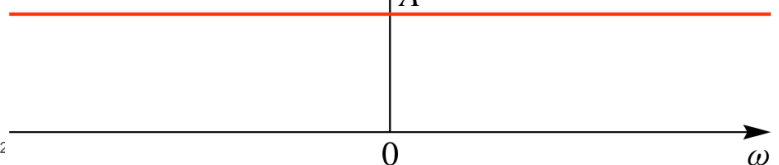
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$$x(t) = \delta(t) \Leftrightarrow X(j\omega) = 1$$

 $X(j\omega)$

A



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0

omega

Example 5:

$$X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)$$

$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

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$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

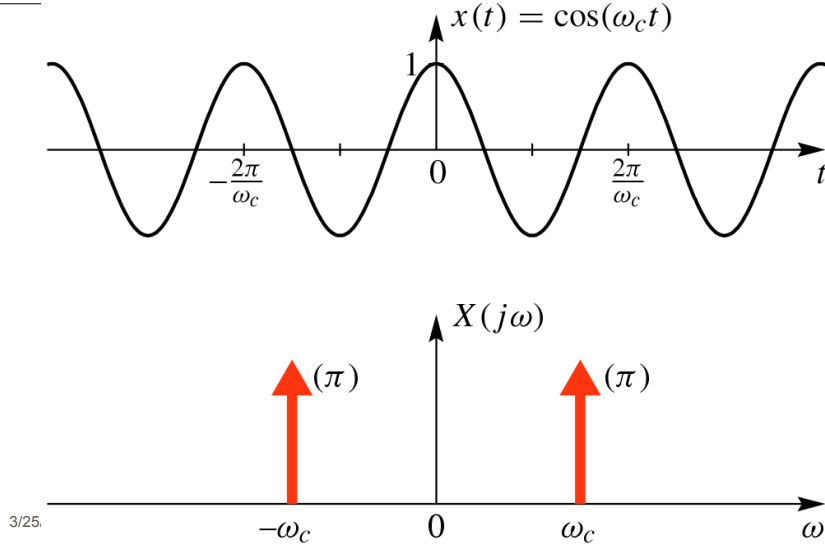


Table of Fourier Transforms

$$x(t) = e^{-at} u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T / 2)}{(\omega / 2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$