

Lecture 22

Sampling and Reconstruction (Fourier View)

5-April-04

Info: Web-CT, Lab, HW

- **Bug in Lab #10 !!!**
 - Change the definition of $x(t)$ in Section 4.3 to be
 - $X(t) = \sin(200\pi t)$
 - Also, Section 4.2 now has some clarification
- **CHECK YOUR GRADES !!!**
 - Web-CT is the OFFICIAL gradebook
- **Quiz #3 will be 9-April (Friday)**
 - Coverage: HW #8, 9, and 10
 - Chapters 9, 10, and 11 (Ch 7 on z-Trans on the Final)
 - Review Session, 8-April, Thurs @ 7:30pm

National Championship Game

- Talent wins games, but teamwork and intelligence win championships.
 - Michael Jordan
- My responsibility is getting all my players playing for the name on the front of the jersey, not the one on the back.
 - Author Unknown
- Mental toughness is to physical as four is to one.
 - Bobby Knight
- Nobody roots for Goliath. - Wilt Chamberlain

LECTURE OBJECTIVES

- **Sampling Theorem Revisited**
 - GENERAL: in the **FREQUENCY DOMAIN**
 - Fourier transform of sampled signal
 - Reconstruction from samples
- Reading: Chap 12, Section 12-3
- Review of FT properties
 - Convolution \leftrightarrow multiplication
 - Frequency shifting
 - Review of AM

Table of FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

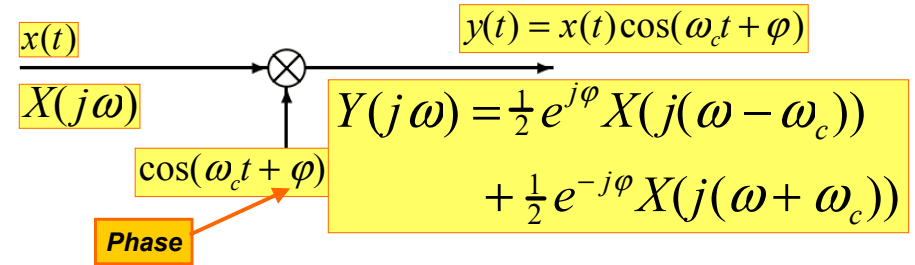
Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\frac{\omega}{a}))$$

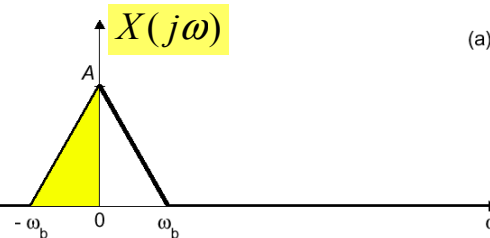
Amplitude Modulator



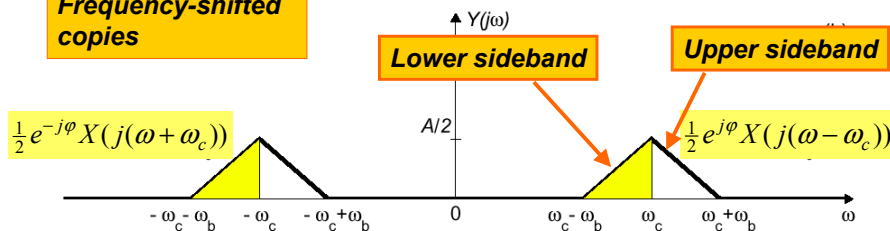
- $x(t)$ **modulates** the amplitude of the cosine wave. The result in the frequency-domain is two **SHIFTED** copies of $X(j\omega)$.

DSBAM: Frequency-Domain

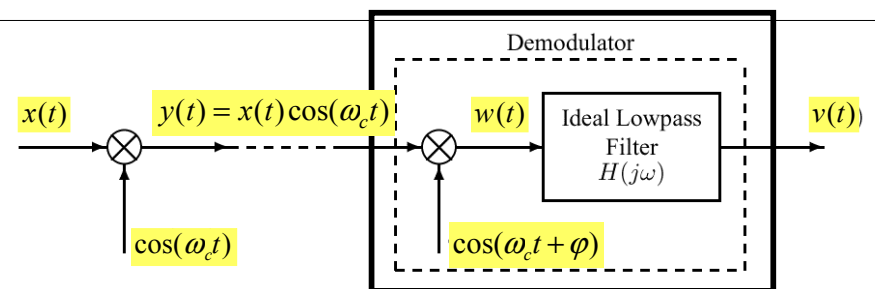
“Typical” bandlimited input signal



Frequency-shifted copies



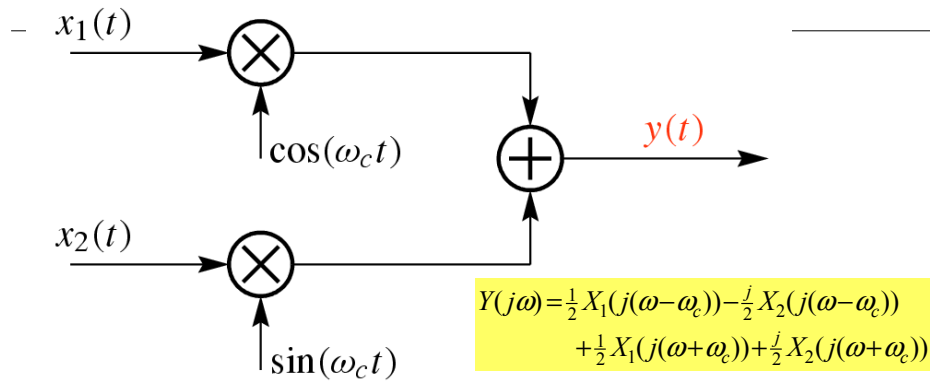
DSBAM Demod Phase Synchron



$$V(j\omega) = \frac{1}{2} \cos(\phi) X(j\omega) \quad \text{what if } \phi = \frac{1}{2}\pi?$$

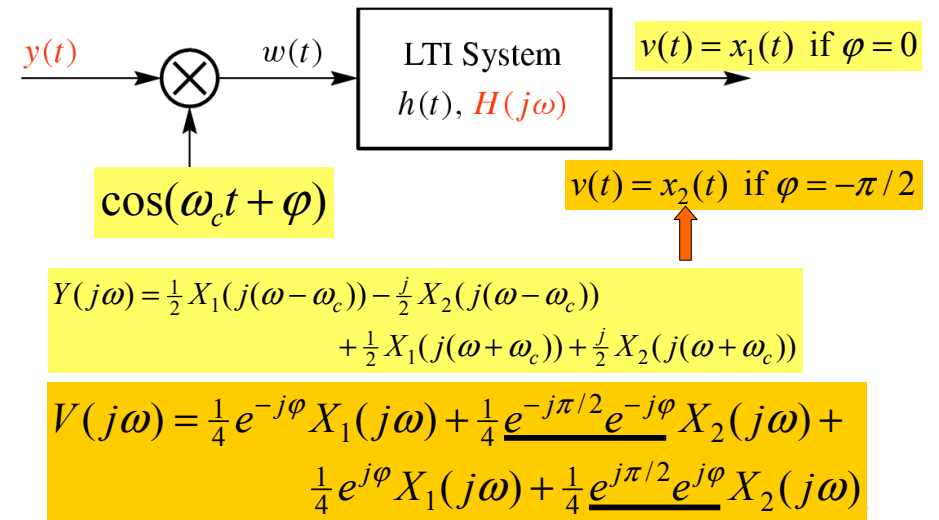
$$W(j\omega) \in \left(\frac{1}{4} e^{j\phi} X(j\omega) + \frac{1}{4} e^{-j\phi} X(j\omega) + \frac{1}{4} e^{j\phi} X(j(\omega - 2\omega_c)) + \frac{1}{4} e^{-j\phi} X(j(\omega + 2\omega_c)) \right)$$

Quadrature Modulator

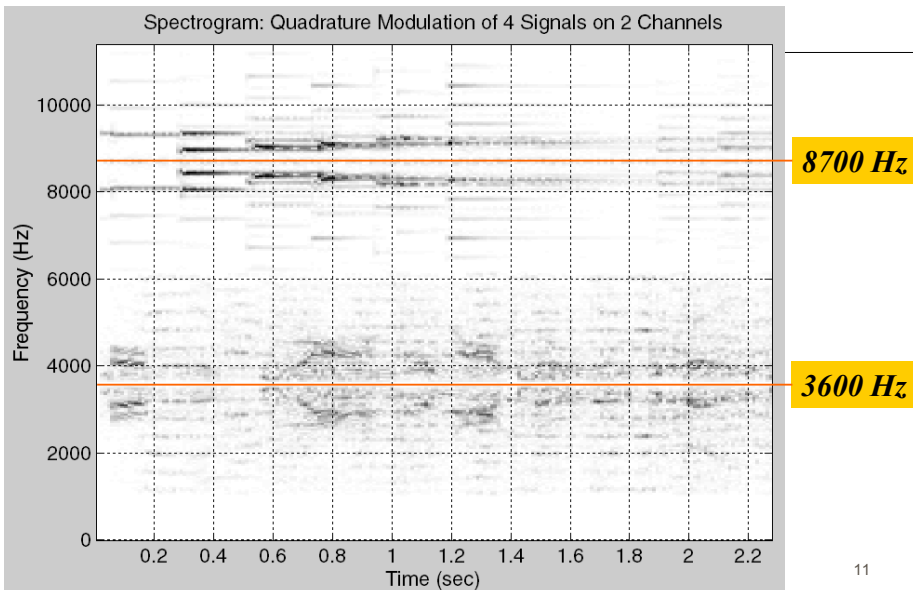


TWO signals on ONE channel: "out of phase"
 Can you "separate" them in the demodulator?

Demod: Quadrature System

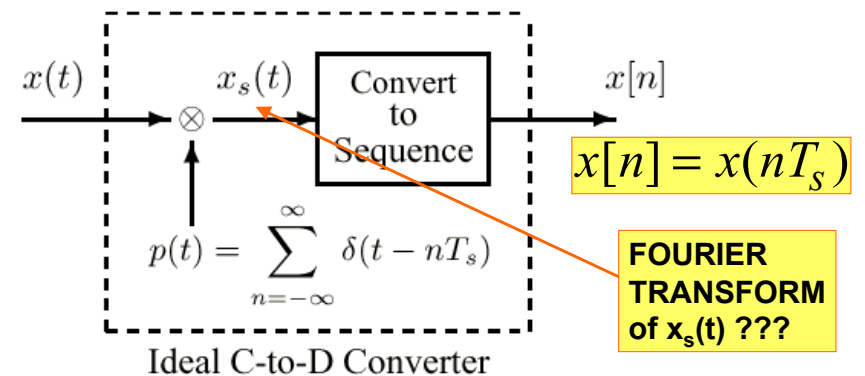


Quadrature Modulation: 4 sigs

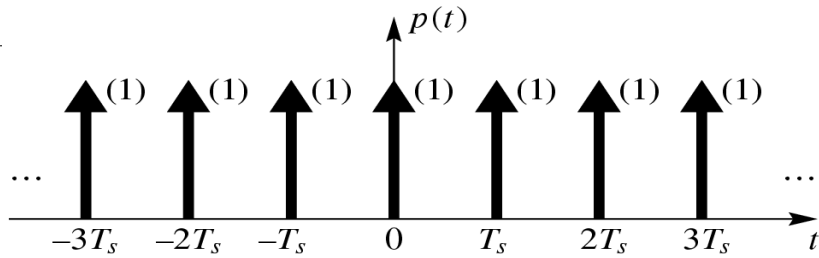


Ideal C-to-D Converter

- Mathematical Model for A-to-D



Periodic Impulse Train



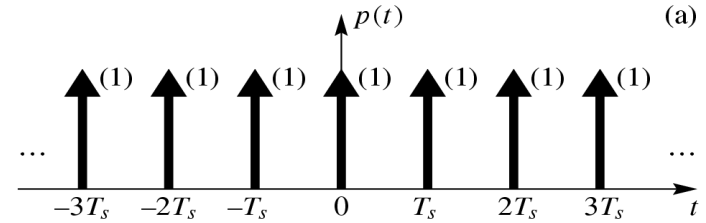
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t} \quad \omega_s = \frac{2\pi}{T_s}$$

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T_s} \quad \text{Fourier Series}$$

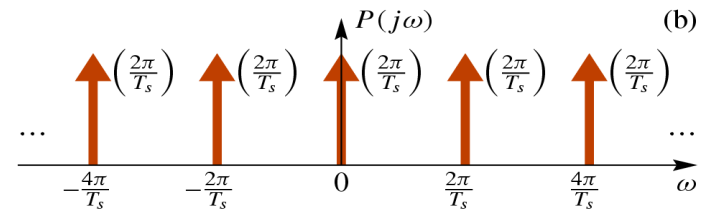
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FT of Impulse Train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \leftrightarrow P(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_s} \delta(\omega - k\omega_s)$$

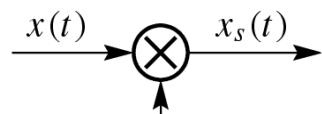


$$\omega_s = \frac{2\pi}{T_s}$$



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Impulse Train Sampling



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

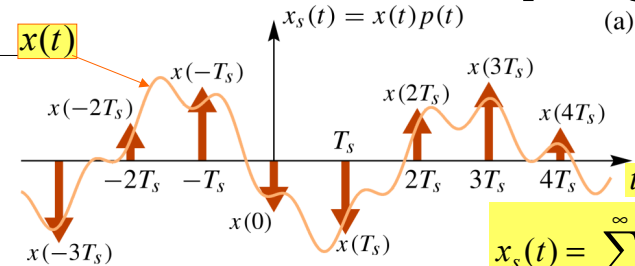
$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s)$$

$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s)$$

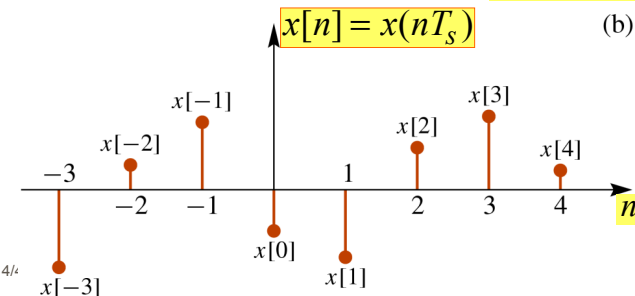
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Illustration of Sampling



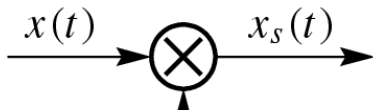
$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s)$$



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Sampling: Freq. Domain



$$p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

**EXPECT
FREQUENCY
SHIFTING !!!**

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

Frequency-Domain Analysis

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s)$$

$$x_s(t) = x(t) \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} x(t) e^{jk\omega_s t}$$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

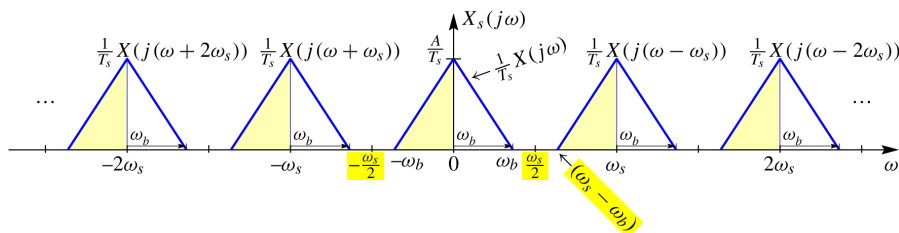
$$\omega_s = \frac{2\pi}{T_s}$$

Frequency-Domain Representation of Sampling

“Typical”
bandlimited signal



$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

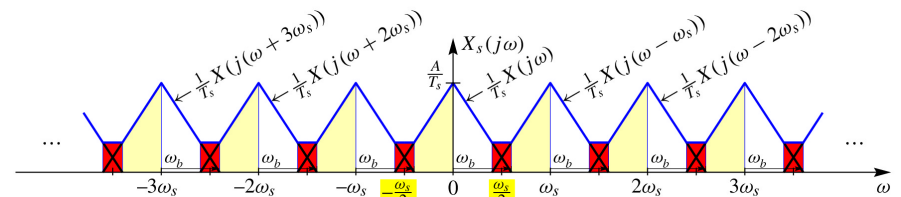


Aliasing Distortion

“Typical”
bandlimited signal



- If $\omega_s < 2\omega_b$, the copies of $X(j\omega)$ overlap, and we have **aliasing distortion**.



Reconstruction of $x(t)$

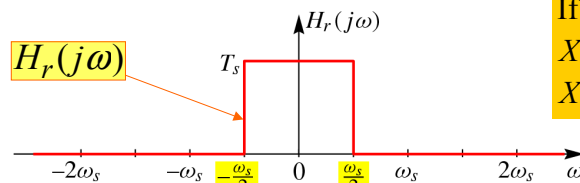
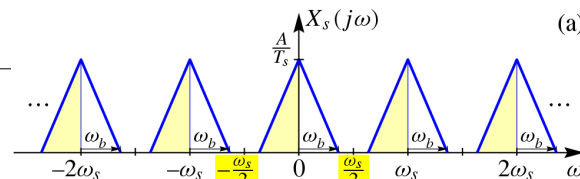


$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

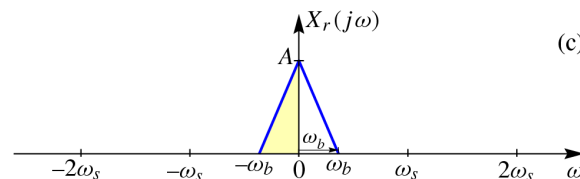
$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega)X_s(j\omega)$$

Reconstruction: Frequency-Domain

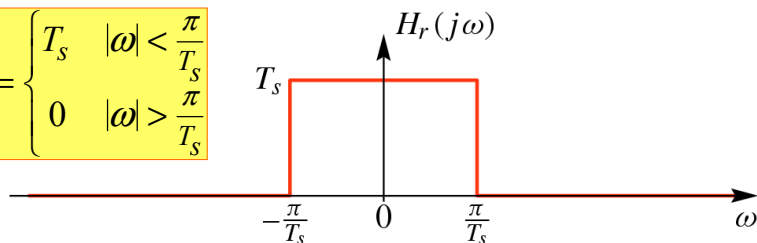


If $\omega_s > 2\omega_b$, the copies of $X(j\omega)$ do not overlap, so $X_r(j\omega) = H_r(j\omega)X_s(j\omega)$

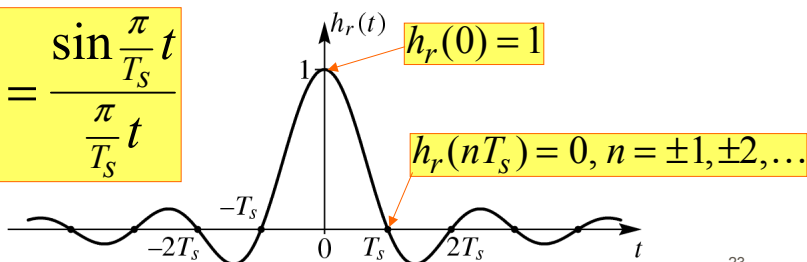


Ideal Reconstruction Filter

$$H_r(j\omega) = \begin{cases} T_s & |\omega| < \frac{\pi}{T_s} \\ 0 & |\omega| > \frac{\pi}{T_s} \end{cases}$$



$$h_r(t) = \frac{\sin \frac{\pi}{T_s} t}{\frac{\pi}{T_s} t}$$



Signal Reconstruction

$$x_r(t) = h_r(t) * x_s(t) = h_r(t) * \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)h_r(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \frac{\pi}{T_s} (t - nT_s)}{\frac{\pi}{T_s} (t - nT_s)}$$

Ideal bandlimited interpolation formula

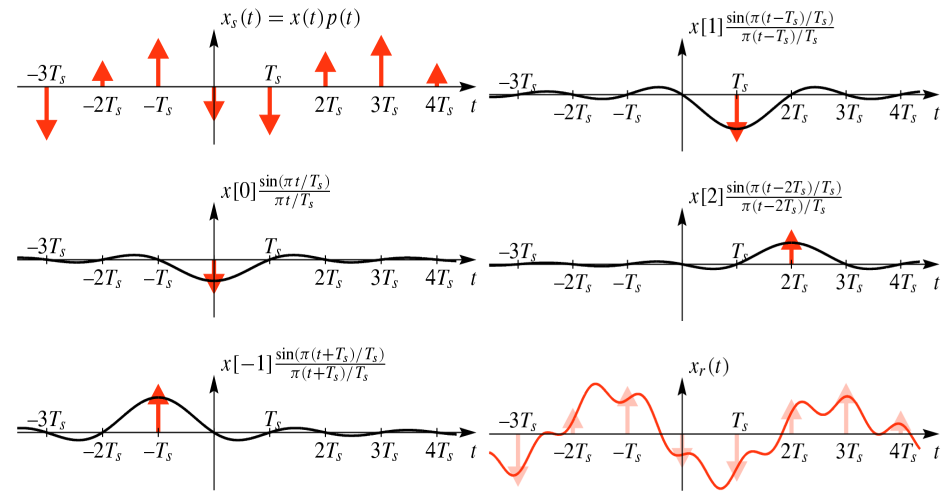
Shannon Sampling Theorem

- **“SINC” Interpolation** is the ideal
 - PERFECT RECONSTRUCTION
 - of BANDLIMITED SIGNALS

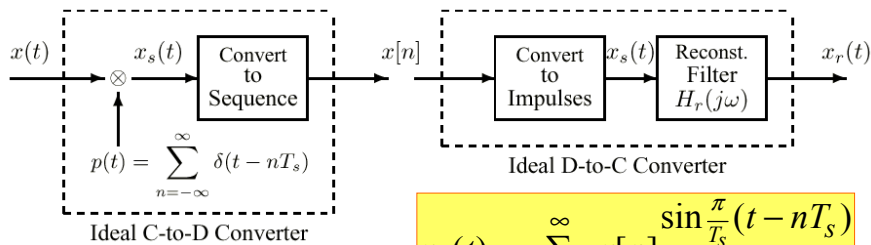
A signal $x(t)$ with bandlimited Fourier transform such that $X(j\omega) = 0$ for $|\omega| \geq \omega_b$ can be reconstructed exactly from samples taken with sampling rate $\omega_s = 2\pi/T_s \geq 2\omega_b$ using the following bandlimited interpolation formula:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \left[\frac{\pi}{T_s} (t - nT_s) \right]}{\frac{\pi}{T_s} (t - nT_s)}.$$

Reconstruction in Time-Domain



Ideal C-to-D and D-to-C



$$x[n] = x(nT_s)$$

Ideal Sampler

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \frac{\pi}{T_s} (t - nT_s)}{\frac{\pi}{T_s} (t - nT_s)}$$

Ideal bandlimited interpolator

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega) X_s(j\omega)$$