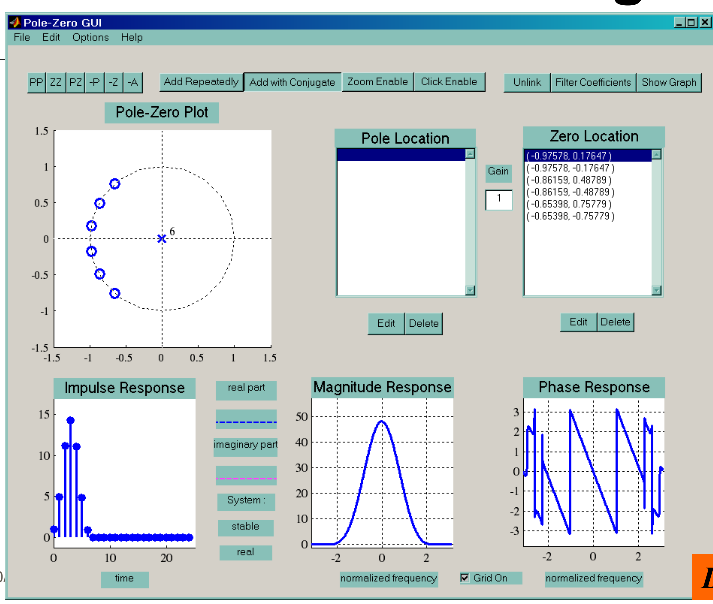
ECE-2025 Spring-04

Lecture 23
IIR Filters: Feedback & H(z)
12-April-04

Info: Web-CT, Lab, HW

- Final Exam on 30-April (Friday, period 15)
  - Coverage: **Everything**
  - One page of **Hand-written** notes (8.5 by 11 in)
- Schedule:
  - Homework:
    - #11 due this week; #12 due the last day of class
  - Labs
    - #11 due the last week
    - PreLab this week; surveys next week
    - #12 will be two parts, entirely in-Lab (i.e., no report)

PeZ Demo: Zero Placing



Lecture

READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, Sects. 8-1, 8-2 & 8-3
- Other Reading:
  - Recitation: Ch. 8, Sects 8-1 thru 8-4
    - POLES & ZEROS
  - Next Lecture: Chapter 8, Sects. 8-4 8-5 & 8-6

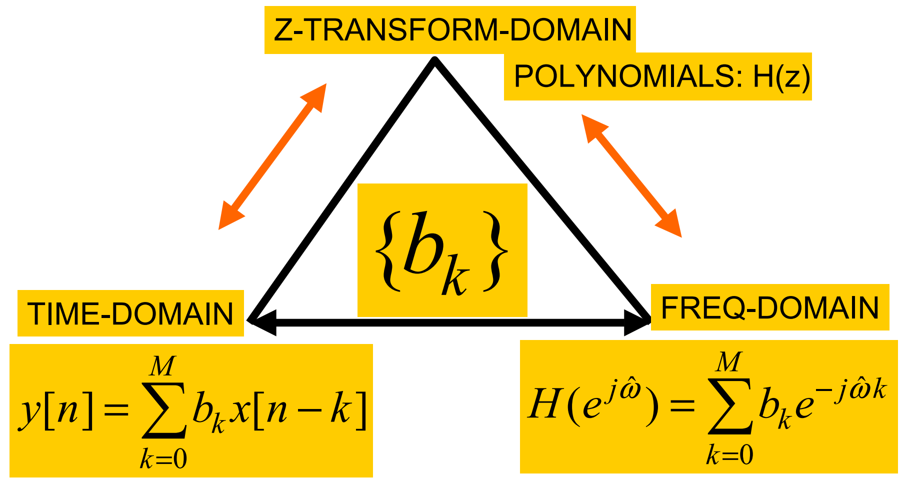
LECTURE OBJECTIVES

- INFINITE IMPULSE RESPONSE FILTERS
  - Define **IIR** DIGITAL Filters
  - Have **FEEDBACK**: use PREVIOUS OUTPUTS

$$y[n] = \sum_{\ell=1}^{N} a_\ell y[n-\ell] + \sum_{k=0}^{M} b_k x[n-k]$$

- Show how to compute the output y[n]
  - FIRST-ORDER CASE (N=1)
  - Z-transform: Impulse Response h[n] ←→ H(z)

THREE DOMAINS



Quick Review: Delay by n_d

$$y[n] = x[n-n_d]$$

IMPULSE RESPONSE $h[n] = \delta[n-n_d]$

SYSTEM FUNCTION $H(z) = z^{-n_d}$

FREQUENCY RESPONSE $H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}n_d}$

Quick Review: L-pt Averager

$$y[n] = \sum_{k=0}^{L-1} \tfrac{1}{L} x[n-k]$$

IMPULSE RESPONSE $h[n] = \sum_{k=0}^{L-1} \tfrac{1}{L} \delta[n-k]$

SYSTEM FUNCTION $H(z) = \sum_{n=0}^{L-1} \tfrac{1}{L} z^{-n}$

LOGICAL THREAD

- FIND the IMPULSE RESPONSE, h[n]
  - INFINITELY LONG
  - **IIR** Filters

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

- EXPLOIT THREE DOMAINS:
  - Show Relationship for IIR:

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

ONE FEEDBACK TERM

- ADD PREVIOUS OUTPUTS

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

PREVIOUS FEEDBACK

FIR PART of the FILTER FEED-FORWARD

- CAUSALITY
  - NOT USING FUTURE OUTPUTS or INPUTS

FILTER COEFFICIENTS

- ADD PREVIOUS OUTPUTS

$$y[n] = 0.8 y[n-1] + 3x[n] - 2x[n-1]$$

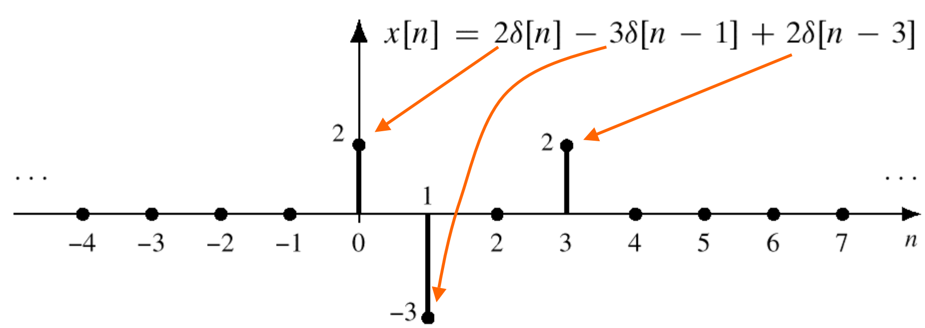
FEEDBACK COEFFICIENT

SIGN CHANGE

- MATLAB
  - `yy = filter([3,-2],[1,-0.8],xx)`

COMPUTE OUTPUT

$$y[n] = 0.8 y[n-1] + 5x[n]$$



COMPUTE y[n]

- FEEDBACK DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 5x[n]$$

- NEED y[-1] to get started

$$y[0] = 0.8y[-1] + 5x[0]$$

AT REST CONDITION

- y[n] = 0, for n<0
- BECAUSE x[n] = 0, for n<0

INITIAL REST CONDITIONS

1. The input must be assumed to be zero prior to some starting time n_0, i.e., $x[n] = 0$ for $n < n_0$. We say that such inputs are *suddenly applied*.
2. The output is likewise assumed to be zero prior to the starting time of the signal, i.e., $y[n] = 0$ for $n < n_0$. We say that the system is *initially at rest* if its output is zero prior to the application of a suddenly applied input.

COMPUTE y[0]

- THIS STARTS THE RECURSION:

With the initial rest assumption, $y[n] = 0$ for $n < 0$,

$$y[0] = 0.8y[-1] + 5(2) = 0.8(0) + 5(2) = 10$$

- SAME with MORE FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]$$

COMPUTE MORE y[n]

- CONTINUE THE RECURSION:

$$y[1] = 0.8y[0] + 5x[1] = 0.8(10) + 5(-3) = -7$$

$$y[2] = 0.8y[1] + 5x[2] = 0.8(-7) + 5(0) = -5.6$$

$$y[3] = 0.8y[2] + 5x[3] = 0.8(-5.6) + 5(2) = 5.52$$

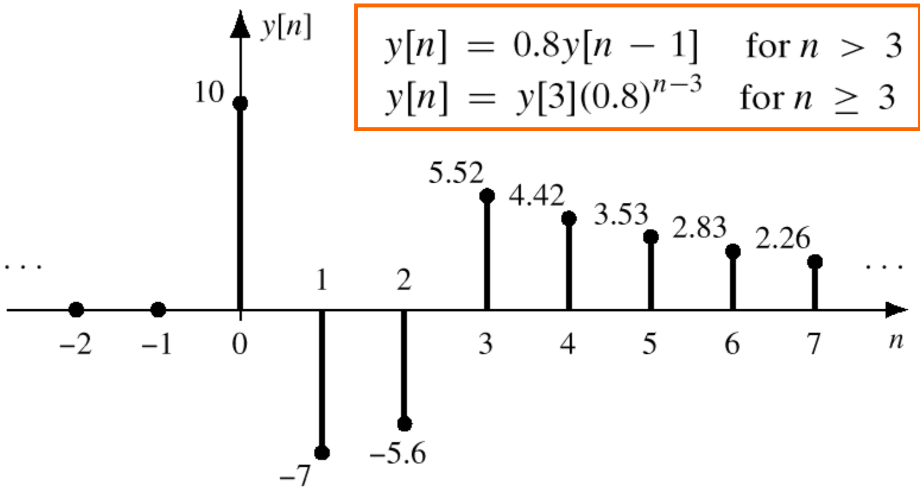
$$y[4] = 0.8y[3] + 5x[4] = 0.8(5.52) + 5(0) = 4.416$$

$$y[5] = 0.8y[4] + 5x[5] = 0.8(4.416) + 5(0) = 3.5328$$

$$y[6] = 0.8y[5] + 5x[6] = 0.8(3.5328) + 5(0) = 2.8262$$

PLOT y[n]

$$y[n] = 0.8y[n-1] \quad \text{for } n > 3$$
$$y[n] = y[3](0.8)^{n-3} \quad \text{for } n \geq 3$$



IMPULSE RESPONSE

$$h[n] = a_1 h[n-1] + b_0\delta[n]$$

n	$n<0$	0	1	2	3	4
$\delta[n]$	0	1	0	0	0	0
$h[n-1]$	0	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$
$h[n]$	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$	$b_0(a_1)^4$

From this table it is obvious that the general formula is

$$h[n] = \begin{cases} b_0(a_1)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$h[n] = b_0(a_1)^n u[n]$$

$$u[n] = 1, \quad \text{for } n \geq 0$$

IMPULSE RESPONSE

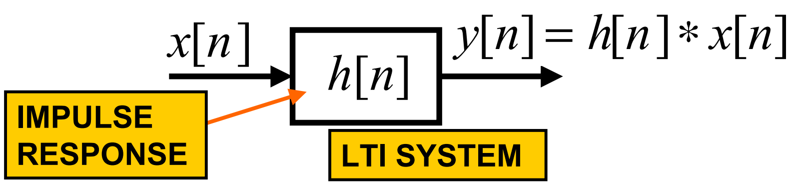
- DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 3x[n]$$

- Find h[n]

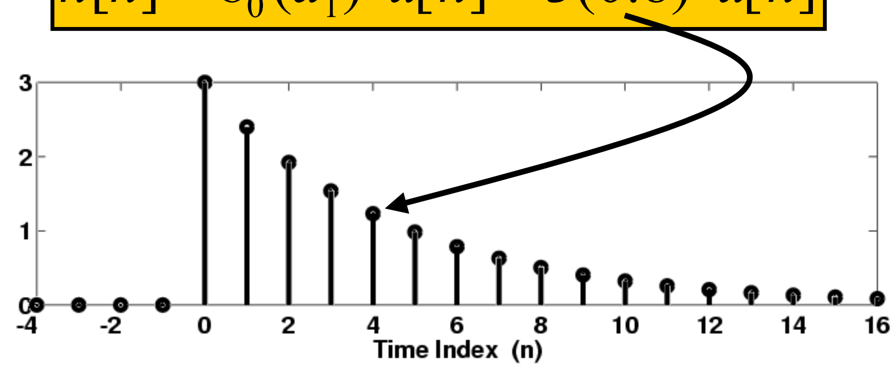
$$h[n] = 3(0.8)^n u[n]$$

- **CONVOLUTION** in TIME-DOMAIN



PLOT IMPULSE RESPONSE

$$h[n] = b_0(a_1)^n u[n] = 3(0.8)^n u[n]$$



Infinite-Length Signal: h[n]

- POLYNOMIAL Representation

$$H(z)=\sum_{n=-\infty}^{\infty} h[n]z^{-n}$$

APPLIES to Any SIGNAL

- SIMPLIFY the SUMMATION

$$H(z)=\sum_{n=-\infty}^{\infty} b_0(a_1)^n u[n]z^{-n} = b_0\sum_{n=0}^{\infty} a_1^n z^{-n}$$

Derivation of H(z)

- Recall Sum of Geometric Sequence:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

- Yields a COMPACT FORM

$$H(z)=b_0\sum_{n=0}^{\infty} a_1^n z^{-n} = b_0\sum_{n=0}^{\infty}(a_1 z^{-1})^n$$

$$=\frac{b_0}{1-a_1 z^{-1}} \quad \text{if } |z|>|a_1|$$

H(z) = z-Transform{ h[n] }

- FIRST-ORDER IIR FILTER:

$$y[n]=a_1 y[n-1]+b_0 x[n]$$

$$h[n]=b_0(a_1)^n u[n]$$

$$H(z)=\frac{b_0}{1-a_1 z^{-1}}$$

H(z) = z-Transform{ h[n] }

- ANOTHER FIRST-ORDER IIR FILTER:

$$y[n]=a_1 y[n-1]+b_0 x[n]+b_1 x[n-1]$$

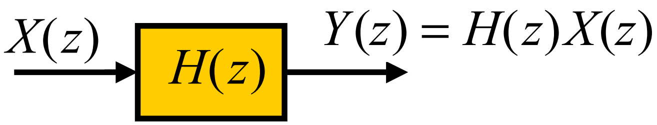
$$h[n]=b_0(a_1)^n u[n]+b_1(a_1)^{n-1}u[n-1]$$

z^{-1} is a shift

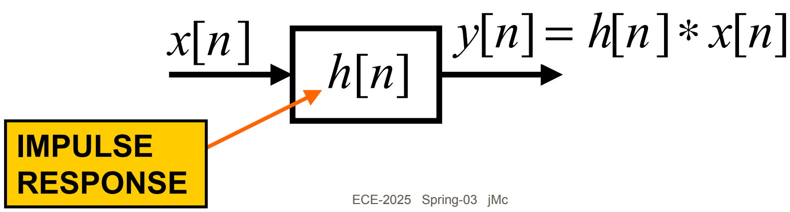
$$H(z)=\frac{b_0}{1-a_1 z^{-1}}+\frac{b_1 z^{-1}}{1-a_1 z^{-1}}=\frac{b_0+b_1 z^{-1}}{1-a_1 z^{-1}}$$

CONVOLUTION PROPERTY

- MULTIPLICATION of z-TRANSFORMS



- CONVOLUTION in TIME-DOMAIN



STEP RESPONSE: x[n]=u[n]

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$u[n] = 1, \quad \text{for } n \geq 0$

n	$x[n]$	$y[n]$
$n < 0$	0	0
0	1	b_0
1	1	$b_0 + b_0(a_1)$
2	1	$b_0 + b_0(a_1) + b_0(a_1)^2$
3	1	$b_0(1 + a_1 + a_1^2 + a_1^3)$
4	1	$b_0(1 + a_1 + a_1^2 + a_1^3 + a_1^4)$
.		.

DERIVE STEP RESPONSE

$$y[n] = b_0(1 + a_1 + a_1^2 + \ldots + a_1^n) = b_0 \sum_{k=0}^{n} a_1^k$$

$$\sum_{k=0}^{L} r^k = \begin{cases} \dfrac{1 - r^{L+1}}{1 - r} & r \neq 1 \\ L + 1 & r = 1 \end{cases}$$

$$y[n] = b_0 \frac{1 - a_1^{n+1}}{1 - a_1} \qquad \text{for } n \geq 0, \qquad \text{if } a_1 \neq 1$$

PLOT STEP RESPONSE