

Lecture 24
H(z) & Frequency Response
for IIR Systems
16-April-04

Info: Web-CT, Lab, HW

▪ **Quiz #3: Results**

- Median = 70
- Quiz #3: Resolve any grade changes by Friday (23-April)

▪ **Total Points posted:**

- Max = 70;
 - percentage is a projection
- 8 Labs, 10 HWs included so far
 - No drops

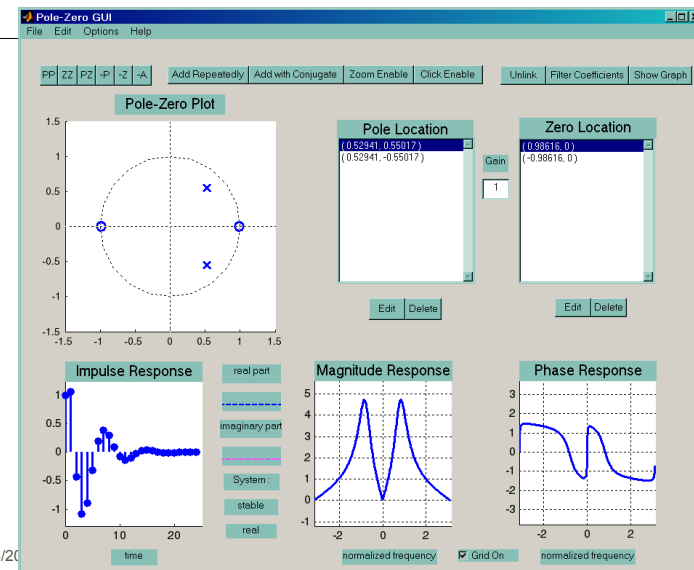
Quiz 3 Histogram

Distribution for Quiz #3

Statistics:
 Graded out of: 100.0 Highest grade: 98.0 Mean grade: 68.2
 Number of records: 166 Lowest grade: 20.0 Median grade: 70.0

Score Range	Frequency
[20, 25)	2
[25, 30)	1
[30, 35)	6
[35, 40)	5
[40, 45)	4
[45, 50)	14
[50, 55)	13
[55, 60)	13
[60, 65)	13
[65, 70)	11
[70, 75)	12
[75, 80)	17
[80, 85)	12
[85, 90)	18
[90, 95)	12
[95, 100)	13

PeZ Demo: Pole-Zero Placing



Lecture

READING ASSIGNMENTS

- This Lecture:
 - Chapter 8, Sects. 8-4 8-5 & 8-6
- Other Reading:
 - Recitation: Chapter 8, all
 - POLE-ZERO PLOTS

LECTURE OBJECTIVES

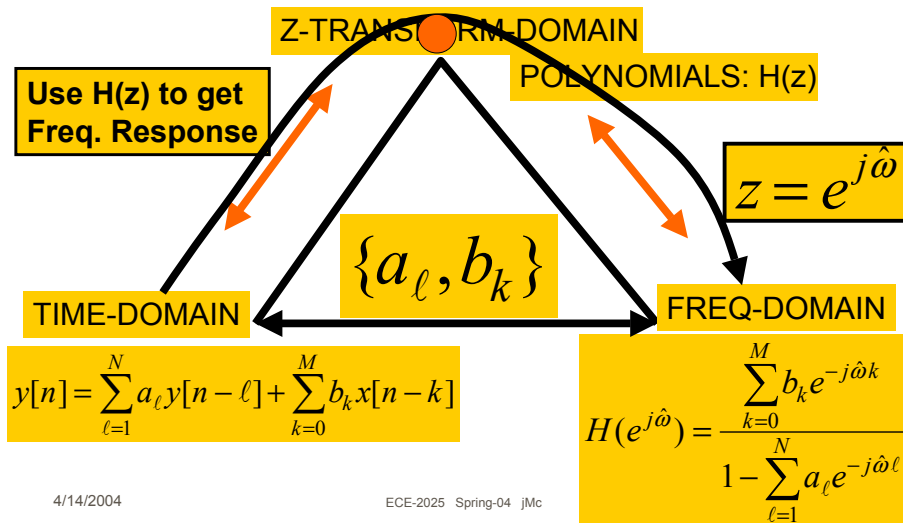
- SYSTEM FUNCTION: $H(z)$
- $H(z)$ has **POLES** and ZEROS
- FREQUENCY RESPONSE of IIR
 - Get $H(z)$ first

$$H(e^{j\hat{\omega}}) = H(z) \Big|_{z=e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

THREE DOMAINS



$H(z) = z$ -Transform{ $h[n]$ }

- FIRST-ORDER IIR FILTER:

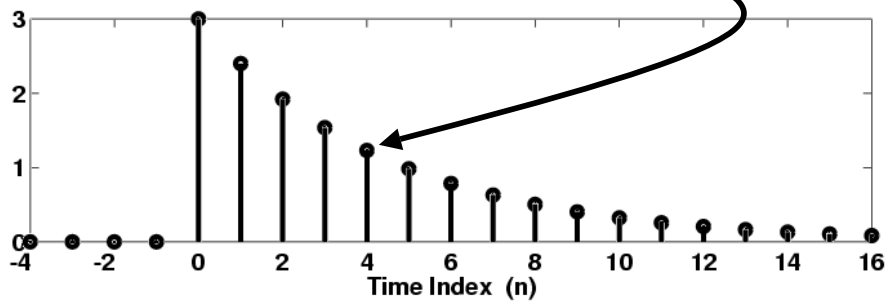
$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$h[n] = b_0 (a_1)^n u[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

Typical IMPULSE Response

$$h[n] = b_0(a_1)^n u[n] = 3(0.8)^n u[n]$$



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First-Order Transform Pair

$$h[n] = ba^n u[n] \leftrightarrow H(z) = \frac{b}{1 - az^{-1}}$$

- GEOMETRIC SEQUENCE:

$$H(z) = b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n$$

$$= \frac{b_0}{1 - a_1 z^{-1}} \quad \text{if } |z| > |a_1|$$

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DELAY PROPERTY of X(z)

- DELAY in TIME \leftrightarrow Multiply X(z) by z^{-1}

$$x[n] \leftrightarrow X(z)$$

$$x[n-1] \leftrightarrow z^{-1} X(z)$$

Proof:
$$\sum_{n=-\infty}^{\infty} x[n-1]z^{-n} = \sum_{\ell=-\infty}^{\infty} x[\ell]z^{-(\ell+1)}$$

$$= z^{-1} \sum_{\ell=-\infty}^{\infty} x[\ell]z^{-\ell} = z^{-1} X(z)$$

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Z-Transform of IIR Filter

- DERIVE the SYSTEM FUNCTION H(z)
 - Use **DELAY PROPERTY**

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)$$

EASIER with DELAY PROPERTY

Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n - n_0] \leftrightarrow z^{-n_0} X(z)$$

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SYSTEM FUNCTION of IIR

- NOTE the FILTER COEFFICIENTS

$$Y(z) - a_1 z^{-1} Y(z) = b_0 X(z) + b_1 z^{-1} X(z)$$

$$(1 - a_1 z^{-1}) Y(z) = (b_0 + b_1 z^{-1}) X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{B(z)}{A(z)}$$

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SYSTEM FUNCTION

- DIFFERENCE EQUATION:

$$y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]$$

- READ** the FILTER COEFFS:

H(z)

$$Y(z) = \left(\frac{3 - 2z^{-1}}{1 - 0.8z^{-1}} \right) X(z)$$

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CONVOLUTION PROPERTY

- MULTIPLICATION** of z-TRANSFORMS

$$X(z) \rightarrow \boxed{H(z)} \rightarrow Y(z) = H(z)X(z)$$

- CONVOLUTION** in TIME-DOMAIN

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = h[n] * x[n]$$

IMPULSE RESPONSE

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POLES & ZEROS

- ROOTS of Numerator & Denominator

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} \rightarrow H(z) = \frac{b_0 z + b_1}{z - a_1}$$

$$b_0 z + b_1 = 0 \Rightarrow z = -\frac{b_1}{b_0}$$

ZERO:
H(z)=0

$$z - a_1 = 0 \Rightarrow z = a_1$$

POLE: H(z) → inf

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EXAMPLE: Poles & Zeros

- VALUE of $H(z)$ at POLES is **INFINITE**

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

$$H(z) = \frac{2 + 2(-1)}{1 - 0.8(-1)} = 0$$

$$H(z) = \frac{2 + 2(\frac{4}{5})^{-1}}{1 - 0.8(\frac{4}{5})^{-1}} = \frac{\frac{9}{2}}{0} \rightarrow \infty$$

ZERO at $z = -1$

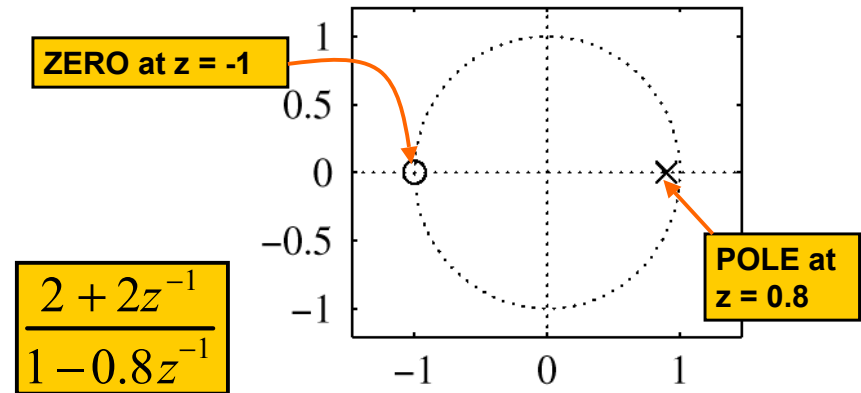
POLE at $z = 0.8$

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POLE-ZERO PLOT



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FREQUENCY RESPONSE

- SYSTEM FUNCTION: $H(z)$
- $H(z)$ has **DENOMINATOR**
- FREQUENCY RESPONSE of IIR
 - We have $H(z)$

$$H(e^{j\hat{\omega}}) = H(z) \Big|_{z=e^{j\hat{\omega}}}$$

- THREE-DOMAIN APPROACH

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

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FREQUENCY RESPONSE

- EVALUATE on the UNIT CIRCLE

$$H(e^{j\hat{\omega}}) = H(z) \Big|_{z=e^{j\hat{\omega}}}$$

$$H(z) = \frac{b_0 + b_1z^{-1}}{1 - a_1z^{-1}}$$

$$H(e^{j\hat{\omega}}) = H(z) \Big|_{z=e^{j\hat{\omega}}} = \frac{b_0 + b_1e^{-j\hat{\omega}}}{1 - a_1e^{-j\hat{\omega}}}$$

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FREQ. RESPONSE FORMULA

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}} \rightarrow H(e^{j\hat{\omega}}) = \frac{2 + 2e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}}$$

$$|H(e^{j\hat{\omega}})|^2 = \left| \frac{2 + 2e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}} \right|^2 = \frac{2 + 2e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}} \cdot \frac{2 + 2e^{j\hat{\omega}}}{1 - 0.8e^{j\hat{\omega}}}$$

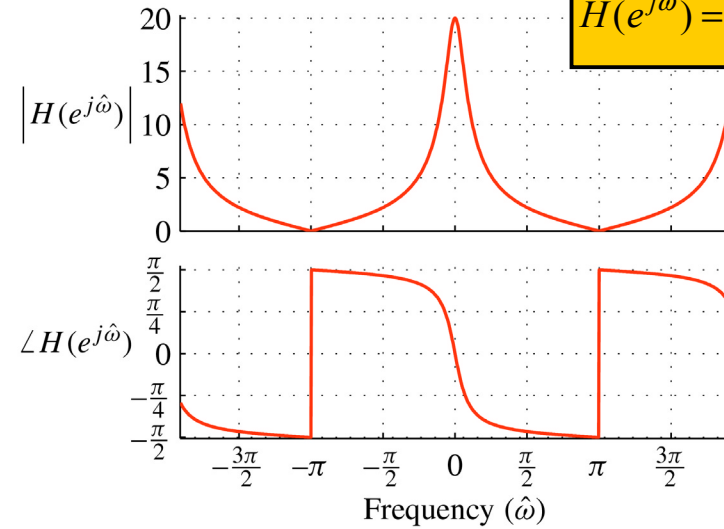
$$\frac{4 + 4 + 4e^{-j\hat{\omega}} + 4e^{j\hat{\omega}}}{1 + 0.64 - 0.8e^{-j\hat{\omega}} - 0.8e^{j\hat{\omega}}} = \frac{8 + 8\cos\hat{\omega}}{1.64 - 1.6\cos\hat{\omega}}$$

$$\text{@ } \hat{\omega} = 0, |H(e^{j\hat{\omega}})|^2 = \frac{8+8}{0.04} = 400, \text{ @ } \hat{\omega} = \pi?$$

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Frequency Response Plot

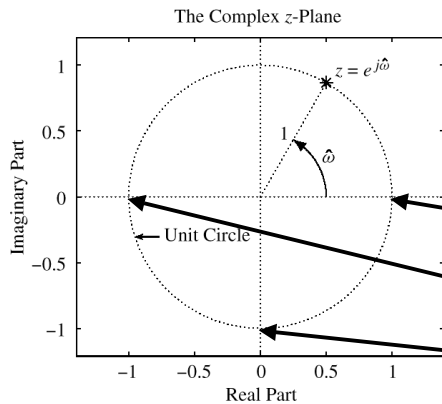
$$H(e^{j\hat{\omega}}) = \frac{2 + 2e^{-j\hat{\omega}}}{1 - 0.8e^{-j\hat{\omega}}}$$



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UNIT CIRCLE

MAPPING BETWEEN z and $\hat{\omega}$



$$z = e^{j\hat{\omega}}$$

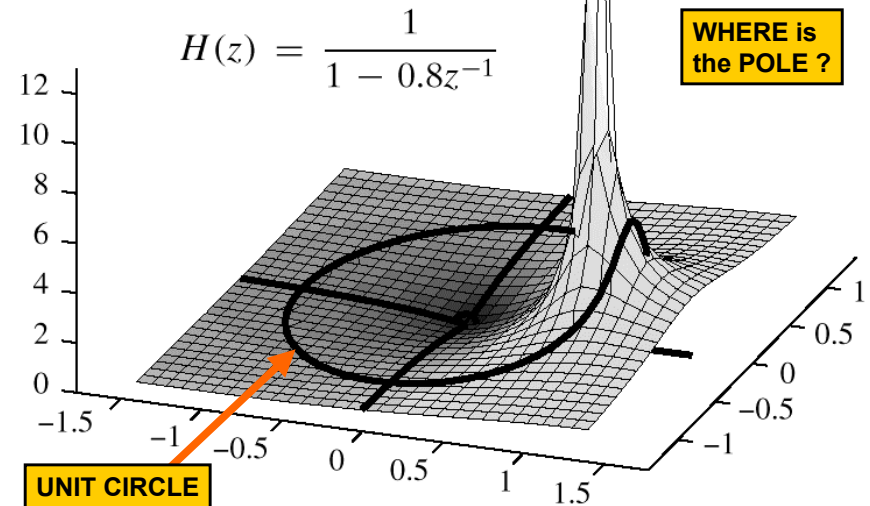
$$\begin{aligned} z = 1 &\leftrightarrow \hat{\omega} = 0 \\ z = -1 &\leftrightarrow \hat{\omega} = \pm\pi \\ z = \pm j &\leftrightarrow \hat{\omega} = \pm\frac{1}{2}\pi \end{aligned}$$

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3-D VIEWPOINT: EVALUTE $H(z)$ EVERYWHERE



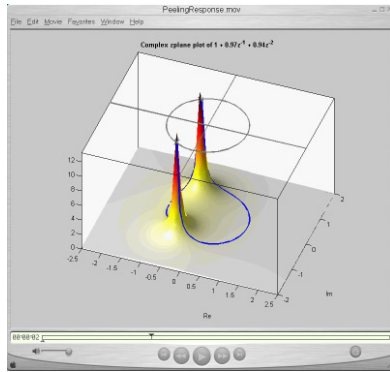
$$H(z) = \frac{1}{1 - 0.8z^{-1}}$$

WHERE is the POLE ?

UNIT CIRCLE

MOVIE for H(z) in 3-D

- POLES to H(z) to Frequency Reponse
 - TWO POLES SHOWN



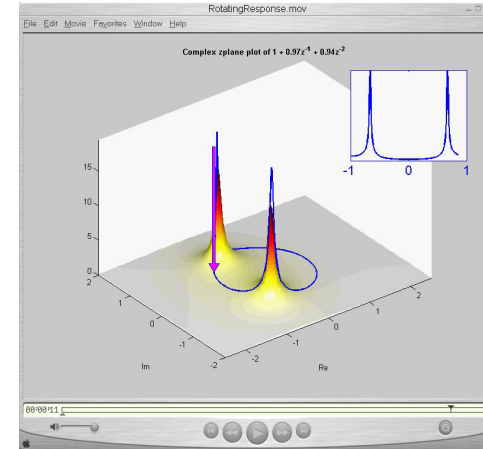
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Frequency Response from H(z)

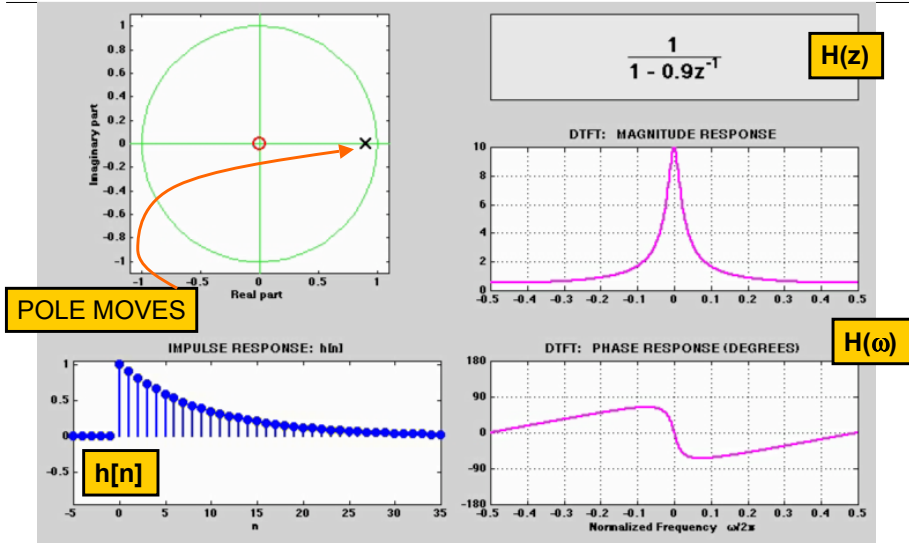
Walking around the Unit Circle



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3 DOMAINS MOVIE: IIR

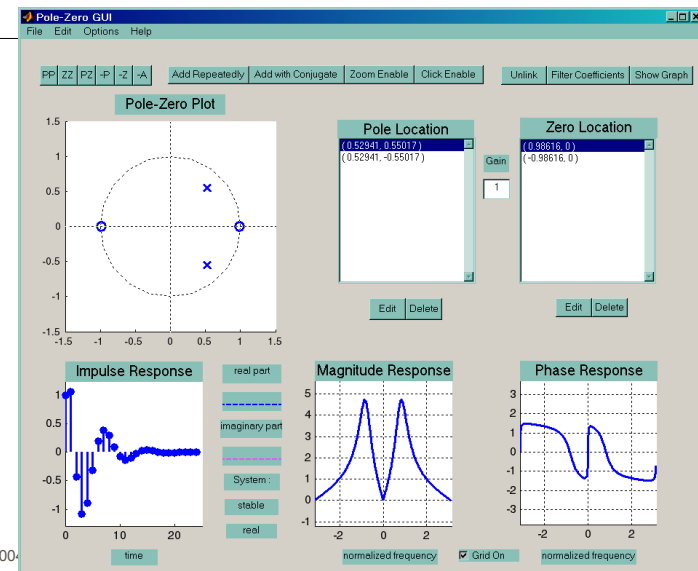


POLE MOVES

$h[n]$

$H(\omega)$

PeZ Demo: Pole-Zero Placing



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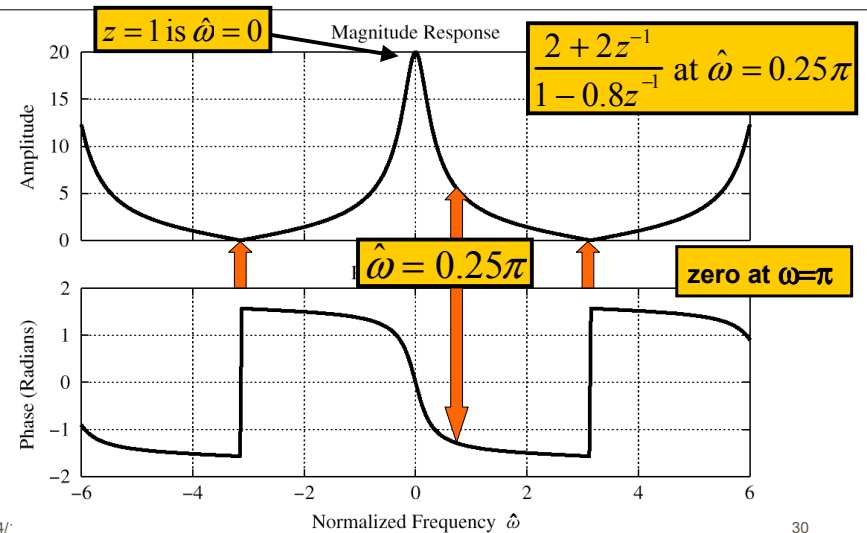
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SINUSOIDAL RESPONSE

- $x[n]$ = SINUSOID \Rightarrow $y[n]$ is SINUSOID
- Get MAGNITUDE & PHASE from $H(z)$

if $x[n] = e^{j\hat{\omega}n}$
 then $y[n] = H(e^{j\hat{\omega}}) e^{j\hat{\omega}n}$
 where $H(e^{j\hat{\omega}}) = H(z)|_{z=e^{j\hat{\omega}}}$

Evaluate FREQ. RESPONSE



POP QUIZ: Eval Freq. Resp.

- Given: $H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$
- Find output, $y[n]$, when $x[n] = \cos(0.25\pi n)$
 - Evaluate at $z = e^{j0.25\pi}$

$$H(z) = \frac{2 + 2(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2})}{1 - 0.8e^{-j0.25\pi}} = 5.182e^{-j1.309}$$

$$y[n] = 5.182 \cos(0.25\pi n - 0.417\pi)$$

CASCADE EQUIVALENT

- Multiply the System Functions

