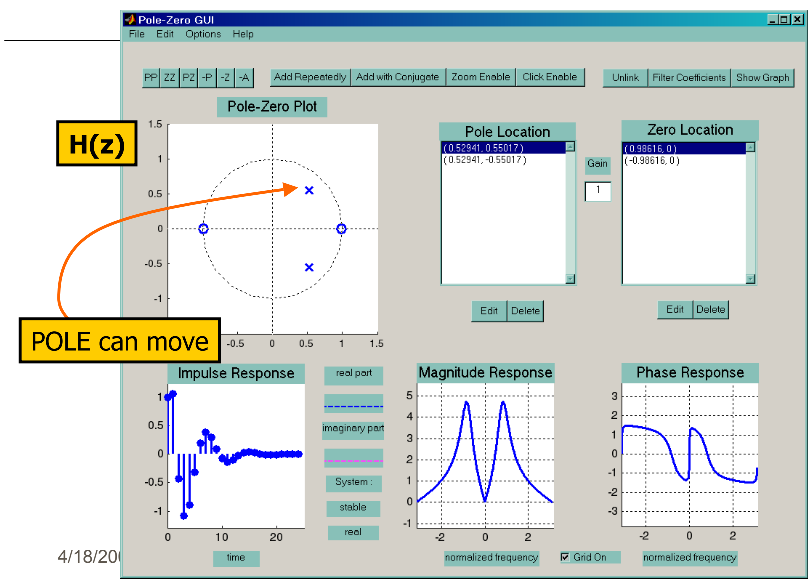
ECE-2025 Spring-2004

Lecture 25
3 Domains for IIR Systems
19-April-04

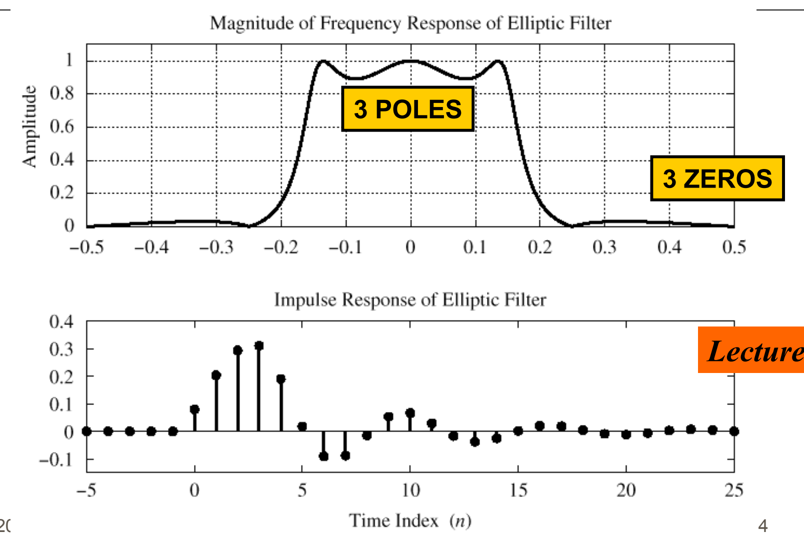
Info: Web-CT, Lab, HW

- Final Exam on 30-April (Friday, period 15)
  - Coverage: **Everything**
  - One page of **Hand-written** notes (8.5 by 11 in)
  - Reviews: Thurs,29-April, 6pm, ECE Auditorium
  - Another one on Wed or Thursday
- Schedule:
  - Homework #12 due the last day of class
    - Lab #11 due the last week
    - Survey this week
    - Lab #12 will be two parts, entirely in-Lab (i.e., no report)

PeZ Demo: Pole-Zero Placing



IIR Elliptic LPF (N=3)



READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, all
- Other Reading:
  - Recitation: Ch. 8, all
    - POLES & ZEROS
  - Next Lecture: Last one

LECTURE OBJECTIVES

- SECOND-ORDER IIR FILTERS
  - TWO FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]$$

- H(z) can have COMPLEX POLES & ZEROS
- THREE-DOMAIN APPROACH
  - BPFs have POLES NEAR THE UNIT CIRCLE

THREE DOMAINS

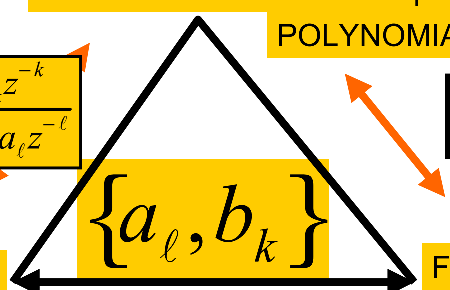
Use H(z) to get Freq. Response

$$H(z) = \frac{\sum b_k z^{-k}}{1 - \sum a_\ell z^{-\ell}}$$

Z-TRANSFORM-DOMAIN: poles & zeros

POLYNOMIALS: H(z)

$$z = e^{j\hat{\omega}}$$



TIME-DOMAIN

$$y[n] = \sum_{\ell=1}^{N} a_\ell y[n-\ell] + \sum_{k=0}^{M} b_k x[n-k]$$

FREQ-DOMAIN

$$H(e^{j\hat{\omega}}) = \frac{\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}}{1 - \sum_{\ell=1}^{N} a_\ell e^{-j\hat{\omega}\ell}}$$

Z-TRANSFORM TABLES

SHORT TABLE OF z-TRANSFORMS

	$x[n]$	$\Longleftrightarrow$	$X(z)$
1.	$ax_1[n] + bx_2[n]$	$\Longleftrightarrow$	$aX_1(z) + bX_2(z)$
2.	$x[n-n_0]$	$\Longleftrightarrow$	$z^{-n_0}X(z)$
3.	$y[n] = x[n] * h[n]$	$\Longleftrightarrow$	$Y(z) = H(z)X(z)$
4.	$\delta[n]$	$\Longleftrightarrow$	1
5.	$\delta[n-n_0]$	$\Longleftrightarrow$	z^{-n_0}
6.	$a^n u[n]$	$\Longleftrightarrow$	$\dfrac{1}{1-az^{-1}}$

SECOND-ORDER FILTERS

- Two FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}$$

MORE POLES

- Denominator is QUADRATIC
  - 2 Poles: REAL
  - or COMPLEX CONJUGATES

$$\frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}$$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 - a_1 z - a_2}$$

PROPERTY OF REAL POLYNOMIALS

A polynomial of degree N has N roots. If all the coefficients of the polynomial are real, the roots either must be real, or must occur in complex conjugate pairs.

TWO COMPLEX POLES

- Find Impulse Response ?
  - Can OSCILLATE vs. n
  - "RESONANCE"

$$(p_k)^n = \left(re^{j\theta}\right)^n = r^n e^{jn\theta}$$

- Find **FREQUENCY RESPONSE**
  - Depends on Pole Location
  - Close to the Unit Circle?
    - Make **BANDPASS FILTER**

$$\text{pole} = re^{j\theta}$$

$$r \rightarrow 1\,?$$

2nd ORDER EXAMPLE

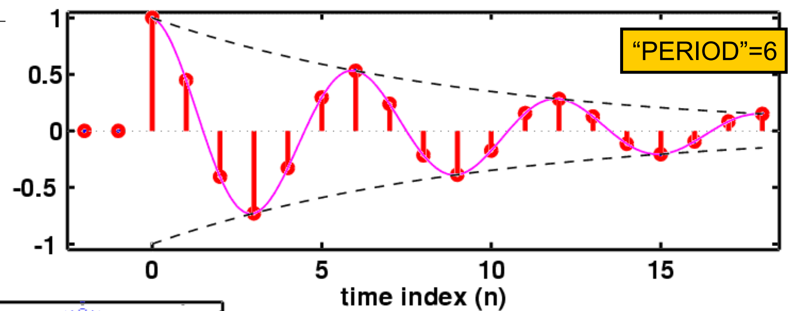
$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3} n) u[n] = (0.9)^n \tfrac{1}{2}(e^{j\pi n/3} + e^{-j\pi n/3}) u[n]$$

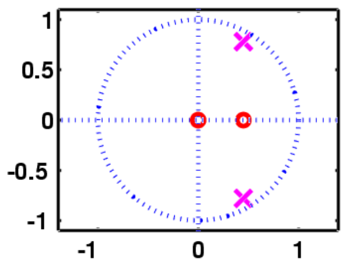
$$H(z) = \frac{0.5}{1 - 0.9e^{j\pi/3} z^{-1}} + \frac{0.5}{1 - 0.9e^{-j\pi/3} z^{-1}}$$

$$H(z) = \frac{1 - 0.9\cos(\frac{\pi}{3}) z^{-1}}{(1 - 0.9e^{j\pi/3} z^{-1})(1 - 0.9e^{-j\pi/3} z^{-1})}$$

$$H(z) = \frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

h[n]: Decays & Oscillates





$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3} n) u[n]$$

$$\frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

2nd ORDER Z-transform PAIR

GENERAL ENTRY for z-Transform TABLE

$$h[n] = r^n \cos(\theta n) u[n]$$

$$H(z) = \frac{1 - r\cos\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

$$h[n] = A r^n \cos(\theta n + \varphi) u[n]$$

$$H(z) = A \frac{\cos\varphi - r\cos(\theta - \varphi) z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

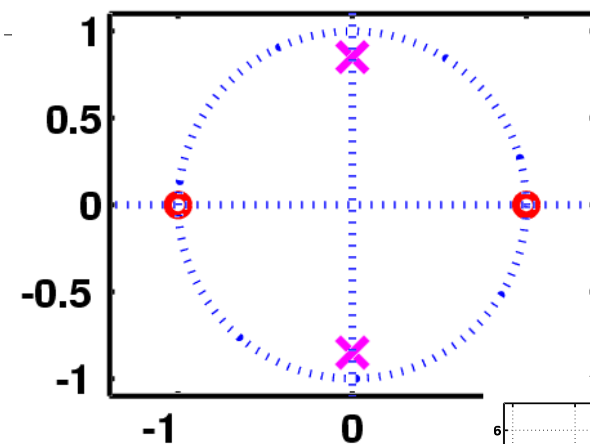
2nd ORDER EX: n-Domain

$$\frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

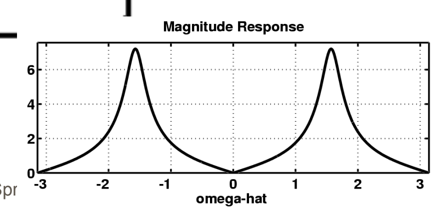
$$y[n] = 0.9 y[n-1] - 0.81 y[n-2] + x[n] - 0.45 x[n-1]$$

```
aa = [ 1, -0.9, 0.81 ];
bb = [ 1, -0.45 ];
nn = -2:19;
hh = filter( bb, aa, (nn==0) );
HH = freqz( bb, aa, [-pi,pi/100:pi] );
```


Complex POLE-ZERO PLOT

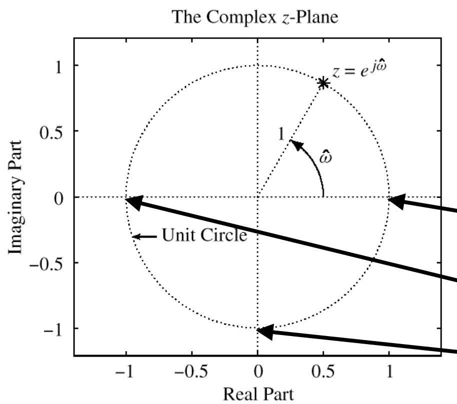


$$\frac{1 - z^{-2}}{1 + 0.7225 z^{-2}}$$



UNIT CIRCLE

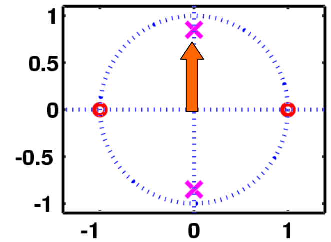
- MAPPING BETWEEN z and $\hat{\omega}$



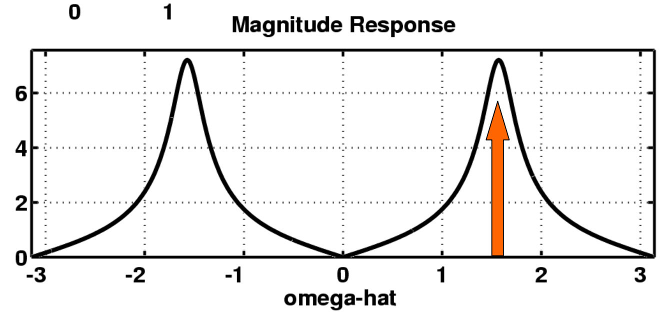
$$z = e^{j\hat{\omega}}$$

$$z = 1 \quad \leftrightarrow \quad \hat{\omega} = 0$$
$$z = -1 \quad \leftrightarrow \quad \hat{\omega} = \pm\pi$$
$$z = \pm j \quad \leftrightarrow \quad \hat{\omega} = \pm\tfrac{1}{2}\pi$$

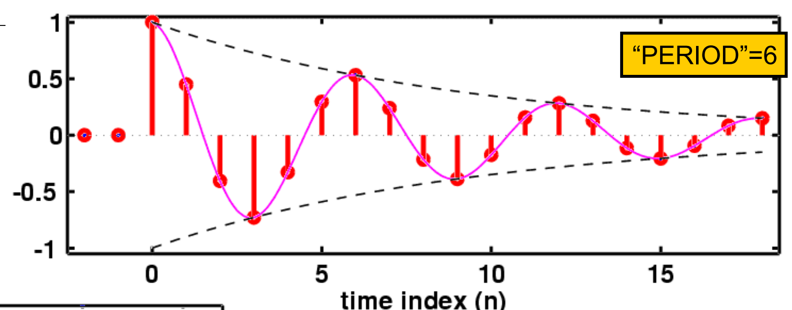
FREQUENCY RESPONSE from POLE-ZERO PLOT

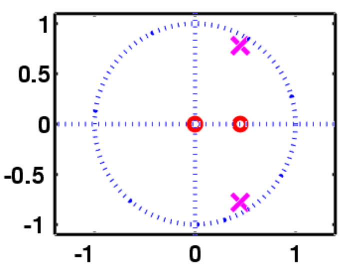


$$H(e^{j\hat{\omega}}) = \frac{1 - e^{-j2\hat{\omega}}}{1 + 0.7225e^{-j2\hat{\omega}}}$$



h[n]: Decays & Oscillates

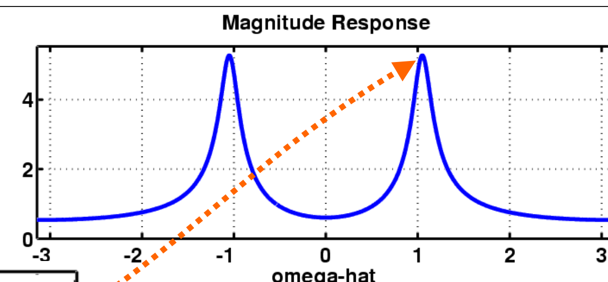


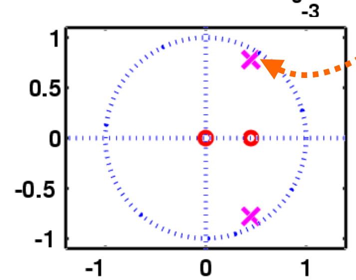


$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3}n)u[n]$$

$$\frac{1 - 0.45z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

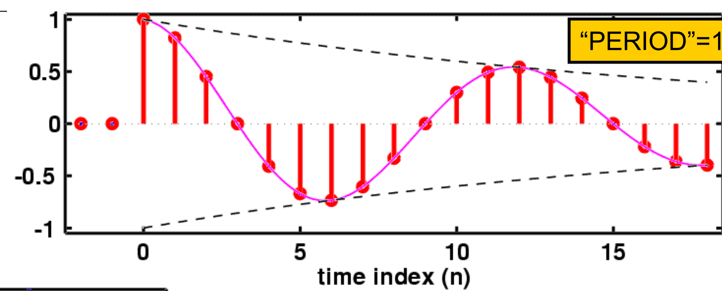
Complex POLE-ZERO PLOT

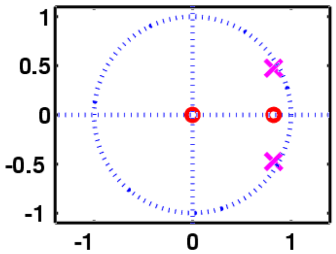




$$\frac{1 - 0.45z^{-1}}{1 - 0.9z^{-1} + 0.81z^{-2}}$$

h[n]: Decays & Oscillates

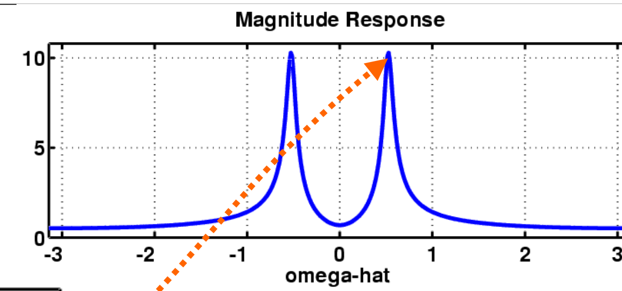


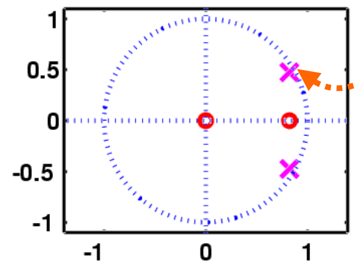


$$h[n] = (0.95)^n \cos(\tfrac{\pi}{6} n) u[n]$$

$$\frac{1 - 0.8227 z^{-1}}{1 - 1.6454 z^{-1} + 0.9025 z^{-2}}$$

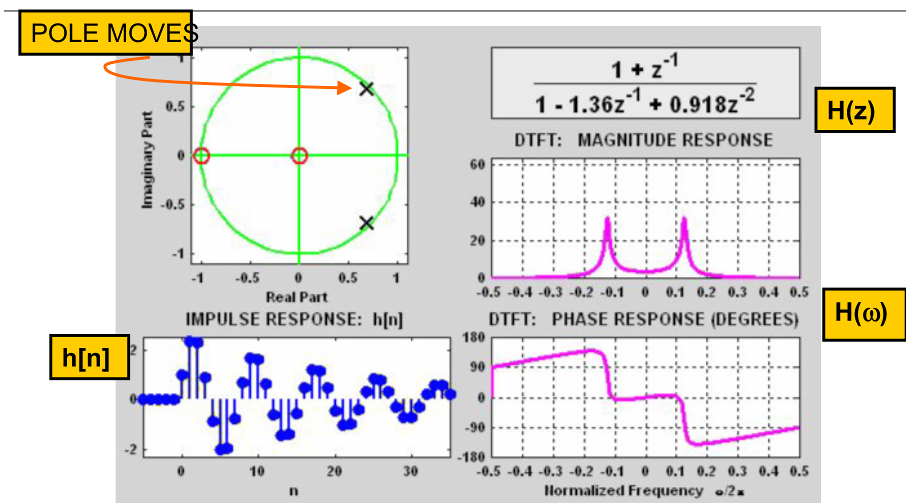
Complex POLE-ZERO PLOT





$$\frac{1 - 0.8227 z^{-1}}{1 - 1.6454 z^{-1} + 0.9025 z^{-2}}$$

3 DOMAINS MOVIE: IIR



THREE INPUTS

- Given: $H(z) = \dfrac{5}{1 + 0.8 z^{-1}}$
- Find the output, y[n]
  - When

$$x[n] = \cos(0.2\pi n)$$

$$x[n] = u[n]$$

$$x[n] = \cos(0.2\pi n) u[n]$$

SINUSOID ANSWER

- Given: $H(z) = \frac{5}{1 + 0.8z^{-1}}$
- The input: $x[n] = \cos(0.2\pi n)$
- Then y[n] $y[n] = M\cos(0.2\pi n + \psi)$

$$H(e^{j0.2\pi}) = \frac{5}{1 + 0.8e^{-j0.2\pi}} = 2.919e^{j0.089\pi}$$

Step Response

$$Y(z) = H(z)X(z) = \left(\frac{5}{1 + .8z^{-1}}\right)\left(\frac{1}{1 - z^{-1}}\right)$$

Partial Fraction Expansion

$$Y(z) = \frac{A}{1 + .8z^{-1}} + \frac{B}{1 - z^{-1}} = \frac{(A + B) + (.8B - A)z^{-1}}{(1 + .8z^{-1})(1 - z^{-1})}$$

$$\Rightarrow \quad (A + B) = 5 \quad \text{and} \quad (.8B - A) = 0$$

$$Y(z) = \frac{A}{1 + .8z^{-1}} + \frac{B}{1 - z^{-1}}$$

Step Response

$$Y(z) = \frac{\frac{20}{9}}{1 + .8z^{-1}} + \frac{\frac{25}{9}}{1 - z^{-1}}$$

$$y[n] = \frac{20}{9}(-.8)^n u[n] + \frac{25}{9}u[n]$$

$$y[n] \to \frac{25}{9} \quad \text{as} \quad n \to \infty$$

Stability

- Nec. & suff. condition: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

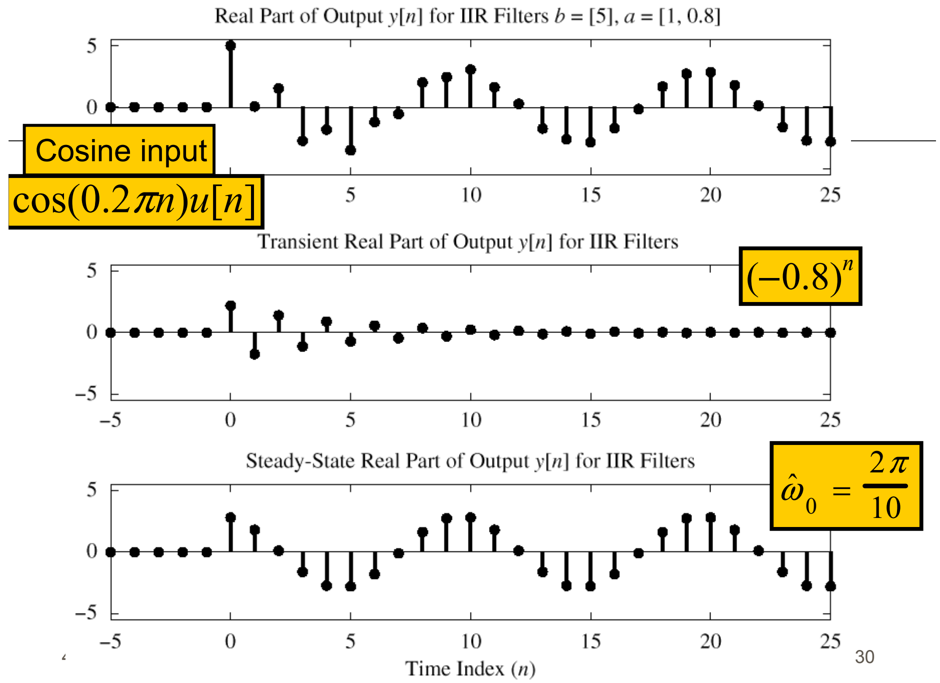
$$h[n] = b(a)^n u[n] \Leftrightarrow H(z) = \frac{b}{1 - az^{-1}}$$

$$\sum_{n=0}^{\infty} |b||a|^n < \infty \text{ if } |a| < 1 \Rightarrow$$

Pole must be Inside unit circle

SINUSOID starting at n=0

- We'll look at an example in MATLAB
  - $\cos(0.2\pi n)$
  - Pole at –0.8, so a^n is $(-0.8)^n$
- There are two components:
  - TRANSIENT
    - Start-up region just after n=0; $(-0.8)^n$
  - STEADY-STATE
    - Eventually, y[n] looks sinusoidal.
    - **Magnitude & Phase from Frequency Response**



STABILITY

- When Does the TRANSIENT DIE OUT ?

STEADY-STATE RESPONSE AND STABILITY

A stable system is one that does not "blow up." This intuitive statement can be formalized by saying that the output of a stable system can always be bounded ($|y[n]| < M_y$) whenever the input is bounded ($|x[n]| < M_x$).[3]

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

$$h[n] = b_0 a_1^n u[n]$$

need $|a_1| < 1$

STABILITY CONDITION

- ALL POLES INSIDE the UNIT CIRCLE
- UNSTABLE EXAMPLE:

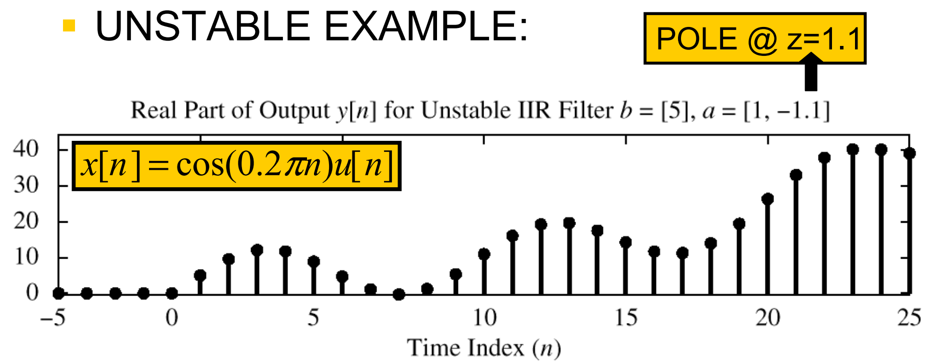


Figure 8.15 Illustration of an unstable IIR system. Pole is at $z = 1.1$.