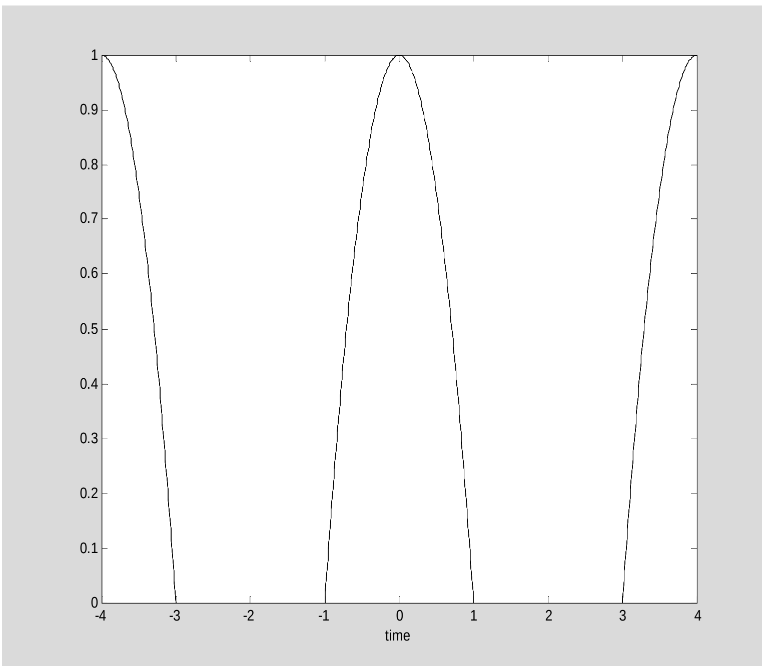
ECE 2025 Homework set #4
Problem 4.1

Part a: First start by defining the signal again, but over a greater time interval. Its periodic, and we know its definition over one period, so all we need do is define it over two more. The definition over all three periods is:

x(t) = 1 - (t-4)^2 for 3 <= t <= 5
x(t) = 1 - (t)^2 for -1 <= t <= 1
x(t) = 1 - (t+4)^2 for -5 <= t <= -3
x(t) = 0 for -3 ≤ t ≤ -1 and 1≤ t ≤ 3 and 5≤ t ≤ 7

The plot of x(t) over the range -4 ≤ t ≤ 4 seconds is then:



Part b: To find the DC value of x(t) we want to determine the value of coefficient a_0. This is found by evaluating:

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$

We can evaluate this over any period of the waveform, and in this case the most convenient will be to use:

$$a_0 = \frac{1}{T_0}\int_{-1}^{3} x(t)dt \quad = \quad \frac{1}{T_0}\int_{-1}^{1} x(t)dt + \frac{1}{T_0}\int_{1}^{3} x(t)dt$$

By inspection of the waveform, x(t)=0 over the time interval 1 to 3 seconds. We also can pick off from the waveform T_0=4. This gives:

$$a_0 = \frac{1}{T_0}\int_{-1}^{1} x(t)dt \quad = \quad \frac{1}{4}\int_{-1}^{1}(1-t^2)dt \quad = \quad \frac{1-\frac{2}{3}}{4} \quad = \quad 0.333$$

Problem 4.2

Part a: The spectrogram shown is one sinusoid being frequency modulated by another. From the spectrogram the signal's frequency is changing between 400 Hz and 700 Hz, and it is changing in a sinusoidal way. Also from the spectrogram, we can see that the frequency is changing at a rate of 4 Hz, in other words the signal transitions between 400 Hz and 700 Hz 4 times per second. To generate such a signal, we need to first determine the instantaneous frequency, ω_i. One way to think about it is that this signal is comprised of a base (or *carrier*) frequency that is being increased or decreased by another frequency component where the rate at which it is increased and decreased by is 4 Hz. Start with the following assignments:

$$\begin{aligned}&\text{Carrier frequency being modulated}: \omega_c\\&\text{Frequency that } \omega_c \text{ is being modulated with}: \omega_{\mathrm{m}}\\&\text{The frequency that determines how fast } \omega_m \text{ changes}: \omega_p\end{aligned}$$

We know from equation 3.56 in the textbook that instantaneous frequency can be found from taking the derivative with respect to time of ψ where $x(t) = A\cos(\psi(t))$.

Using these definitions to express instantaneous frequency could give as an example:

$$\omega_i = (\omega_m * \sin(\omega_p t + \phi_p)) + \omega_c$$

This seems reasonable. Intuitively it suggests that we will add to a carrier frequency (ω_c) a modulating frequency (ω_m). This modulating frequency varies sinusoidally at a rate given by ω_p. We know from equation 3.43 in the textbook that frequency modulated signals are described by a class of signals with time-varying angle functions of the form $x(t) = A\cos(\psi(t))$. Now that we have an expression for instantaneous frequency we can find our angle function ψ by integrating ωi over time.

$$\begin{aligned}\psi(t) &= \int_0^t \omega_m \sin(\omega_p t + \phi_p) + \omega_c \text{ dt}\\&= -\frac{\omega_{\mathrm{m}}}{\omega_{\mathrm{p}}}\cos(\omega_p t + \phi_p) + \omega_c t + \phi_c\end{aligned}$$

From the spectrogram we determined that the average frequency of the signal is 550Hz, so we set ω_c to $2\pi550$. The frequency that ω_c is being modulated with (ω_m) is +- $2\pi150$ Hz giving the maximum and minimum frequencies of the output signal of 700Hz and 400 Hz respectively, so ω_m is $2\pi150$. ω_m transitions between $+2\pi150$Hz and $-2\pi150$Hz four times per second, so ω_p is $2\pi4$. Making these substitutions gives:

$$\psi(t) = -\frac{2\pi150}{2\pi4}\cos(2\pi4t + \phi_p) + 2\pi550t + \phi_c$$

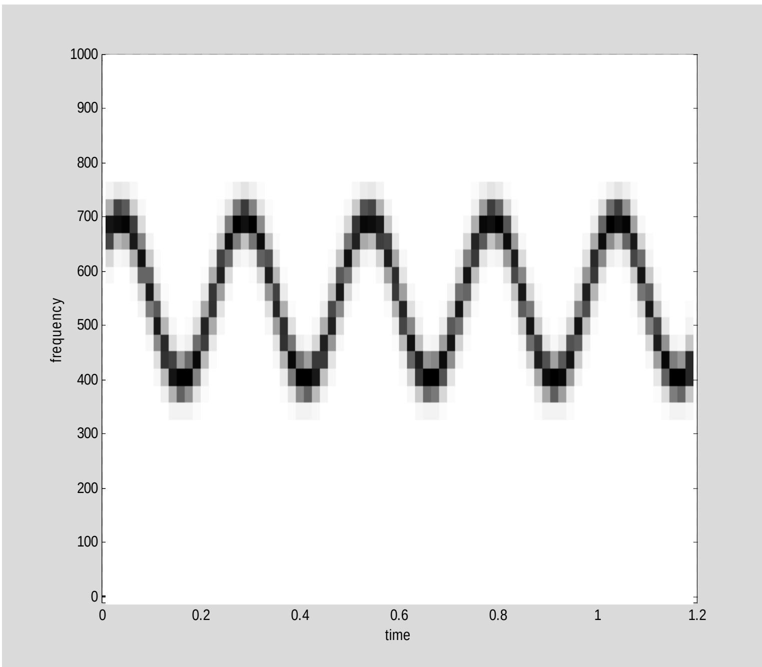
Now we have the angle function ψ, we can use textbook equation 3.43 which describes the general form of signals that use time-varying angle functions. Using our equation for ψ with equation 3.43 gives:

$$x(t) = A\cos(\psi(t)) = A\cos\left(-\frac{150}{4}\cos(2\pi4t + \phi_p) + 2\pi550t + \phi_c\right)$$

The only thing left to do is determine the modulating phase angle, φ_p. Our modulating signal is a sin function, and using the spectrogram we determine by inspection a phase angle of $\pi/5$. The parameters A and φ_c are not determinable from the spectrogram, and in fact you can put in almost any value you'd like. Using all the values determined so far, one possible signal that could produce the spectrogram is:

$$x(t) = A\cos(\psi(t)) = 50\cos\left(-\frac{150}{4}\cos(2\pi 4t + \pi/5) + 2\pi 550t + 0\right)$$

Although not specifically asked for in this problem, one good way to check your result is to evaluate it in MATLAB and plot the resulting spectrogram. The spectrogram of the above expression for x(t) gives:



Part b: As noted in part a, there are two parameters out of the six that cannot be determined uniquely from the spectrogram. They are parameters A and F. The parameter A is the amplitude of x(t) and parameter F is its phase (φ_c). Because the spectrogram shows frequency versus time and not signal amplitude versus time, we cannot determine these parameters from it.

Problem 4.3
Part a:

Note name	G	G$^\sharp$	A	B$^\flat$	B	C	C$^\sharp$	D	E$^\flat$	E	F	F$^\sharp$	G
Note number	47	48	49	50	51	52	53	54	55	56	57	58	59
Frequency (Hz)	392	415	440	466	494	523	554	587	622	659	698	740	784

Part b:

$$f = 440 * f^{\left(\frac{note\_number-49}{12}\right)} \text{ Hz}$$

Problem 4.4
Part a:

The period of the signal is determined from the greatest common divisor of the frequencies shown in the spectrum:

GCD(30π,80π) = 10πω = 5 Hz   From this the period is just 1/5Hz=0.2 seconds.

Part b:

The DC value of this signal is equal to 5. This is the amplitude shown on the spectrum at ω=0.

Problem 4.5

Part a: We want to use the relation y(t)=3x(t) - 2 and express the Fourier series coefficients b_k for the signal y(t) in terms of the coefficients a_k for the signal x(t). We start by expressing x(t) and y(t) in terms of their Fourier coefficients as:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jw_0 kt}$$
$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jw_0 kt}$$

We know that y(t) = 3x(t) - 2. Substituting for x(t) gives y(t) expressed in terms of a_k instead of b_k:

$$y(t) = 3\left(\sum_{k=-\infty}^{\infty} a_k e^{jw_0 kt}\right) - 2 \qquad \text{or}$$
$$\text{y(t)} = -2 + \left(\sum_{k=-\infty}^{\infty} 3a_k e^{jw_0 kt}\right)$$

Part b: This is a similar problem to part a, except you now have z(t)=x(t+1). z(t) is a signal with Fourier coefficients c_k, and we want to express z(t) in terms of the a_k coefficients belonging to signal x(t). Again we start with:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jw_0 kt}$$
$$z(t) = \sum_{k=-\infty}^{\infty} c_k e^{jw_0 kt}$$

We know that z(t) = x(t+1). All we need to do is determine the signal x(t+1):

$$x(t+1) = \sum_{k=-\infty}^{\infty} a_k e^{jw_0 k(t+1)} = \sum_{k=-\infty}^{\infty} a_k e^{jw_0 kt} e^{jw_0 k}$$

Now substituting in for x(t-1) allows us to express the coefficients c_k in terms of a_k:

$$z(t) = \sum_{k=-\infty}^{\infty} \left(a_k e^{jw_0 k}\right) e^{jw_0 kt}$$