ECE-2025 Spring-2005

Lecture 6

Fourier Series Analysis

31-Jan-2004

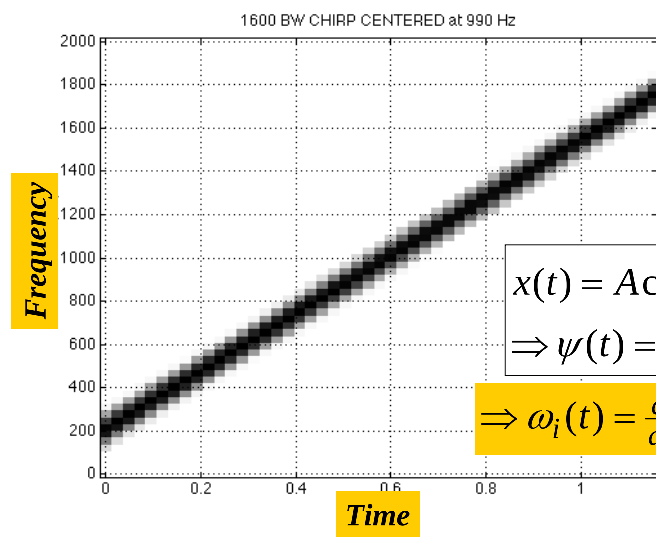
General Info

- ***Help Sessions: M6, T 6 and W6***
  - ***Office Hours:*** Visit any Prof or TA
  - ***Bulletin Board: OFFICIAL ANNOUNCEMENTS***
- ***Quiz #1 on 4-Feb***
  - Coverage: HW #1, #2, and #3
  - Old Quizzes & Problems are linked via WebCT
  - Review on Thursday evening: time = ??? 6 or 7 pm.
- Lab #3: **bring headphones**
  - Lots to read about chirps
- Lab #2 report due at beginning of lab
  - **Ask your grading TA about his/her format**
- Prob Set #3 due THIS Week
  - Prob Set #4 (easy) will be due during week of 7-Feb (plan ahead)

The Rules

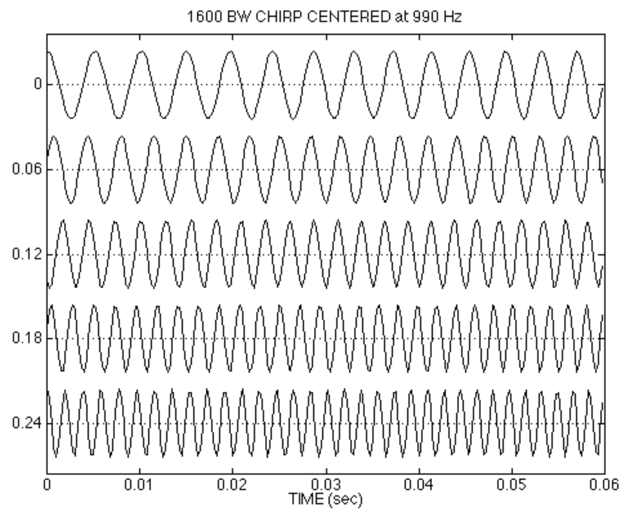
- Quizzes
  - NO make-ups given
  - Next Quiz would count for the one missed, IF excused
- Excused Absence
  - Must be written (by an "official")
  - Notify ahead of time via e-mail
- Consult "INFO" on Web-CT for more details
  - Late Labs are –10 points per day
  - No late Homework

CHIRP SPECTROGRAM



$$x(t) = A\cos(\alpha t^2 + \beta t + \varphi)$$
$$\Rightarrow \psi(t) = \alpha t^2 + \beta t + \varphi$$
$$\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t) = 2\alpha t + \beta$$

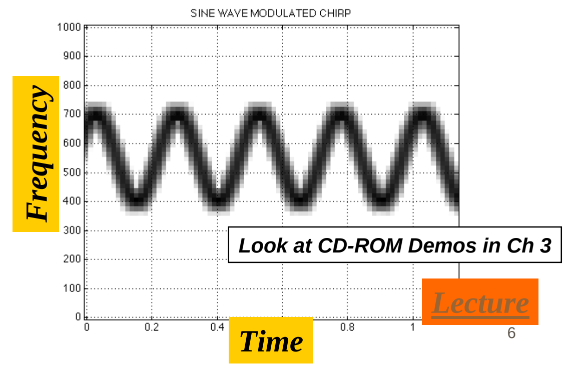
CHIRP WAVEFORM



SINE-WAVE FREQUENCY MODULATION (FM)

$$x(t) = A\cos(\alpha \sin(\beta t + \psi) + \varphi)$$

$$\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t) = \alpha\beta\cos(\beta t + \psi)$$



READING ASSIGNMENTS

- This Lecture:
  - **Fourier Series in Ch 3, Sects 3-4, 3-5 & 3-6**
    - Replaces pp. 62-66 in Ch 3 in DSP First
    - Notation: a_k for Fourier Series
- Other Reading:
  - Next Lecture: More Fourier Series

LECTURE OBJECTIVES

- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi k/T_0)t} dt$$

- **ANALYSIS** via Fourier Series
  - For **PERIODIC** signals: $x(t+T_0) = x(t)$
  - Later: spectrum from the Fourier Series

HISTORY

- Jean Baptiste Joseph Fourier
  - 1807 thesis (memoir)
    - On the Propagation of Heat in Solid Bodies
  - Heat !
  - Napoleonic era
- http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html

Joseph Fourier

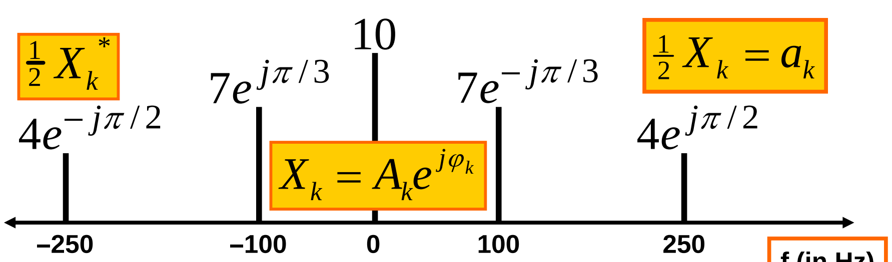
lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Find out more at:
http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fourier.html

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = a_0 + \sum_{k=1}^{N} \{ a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \}$$

Harmonic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

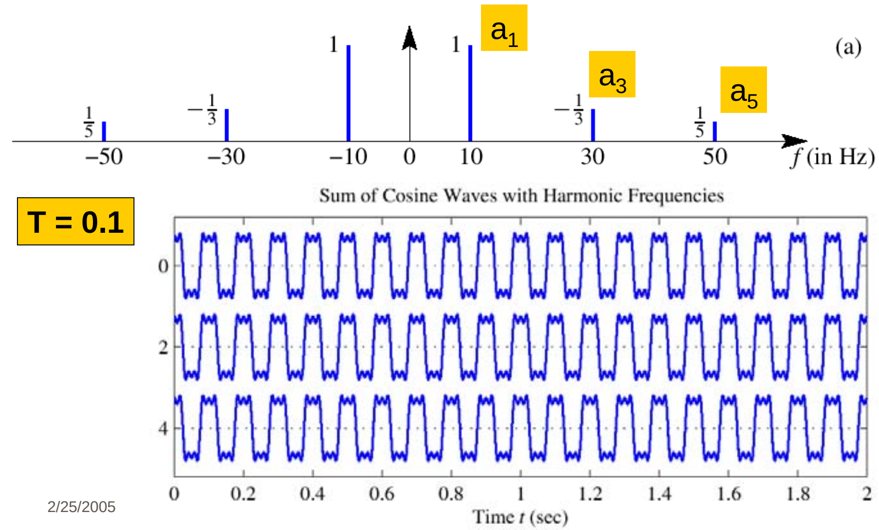
$$a_k = \tfrac{1}{2} X_k = \tfrac{1}{2} A_k e^{j\varphi_k}$$

$$x(t) = a_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

COMPLEX AMPLITUDE

$$X_k = A_k e^{j\varphi_k}$$

Harmonic Signal (3 Freqs)



SYNTHESIS vs. ANALYSIS

- SYNTHESIS
  - **Easy**
  - Given (ω_k, A_k, ϕ_k) create x(t)
- Synthesis can be HARD
  - Synthesize Speech so that it sounds good
- ANALYSIS
  - **Hard**
  - Given x(t), extract (ω_k, A_k, ϕ_k)
  - How many?
  - Need algorithm for computer

STRATEGY: x(t) → a_k

- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

INTEGRAL Property of exp(j)

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}\,dt = \frac{T_0}{-j2\pi m} e^{-j(2\pi/T_0)mt}\Big|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m}\left(e^{-j2\pi m} - 1\right)$$

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}\,dt = 0$$

$m \neq 0$)

$$\omega_0 = \frac{2\pi}{T_0}$$

ORTHOGONALITY of exp(j)

- PRODUCT of exp(+j) and exp(-j)

$$\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)\ell t} e^{-j(2\pi/T_0)kt}\,dt = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$$

$$\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)(\ell-k)t}\,dt$$

Isolate One FS Coefficient

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}$$

$$\frac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi/T_0)\ell t}\,dt = \frac{1}{T_0}\int_0^{T_0}\left(\sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}\right) e^{-j(2\pi/T_0)\ell t}\,dt$$

$$\frac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi/T_0)\ell t}\,dt = \sum_{k=-\infty}^{\infty} a_k \left(\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)kt} e^{-j(2\pi/T_0)\ell t}\,dt\right) = a_\ell$$

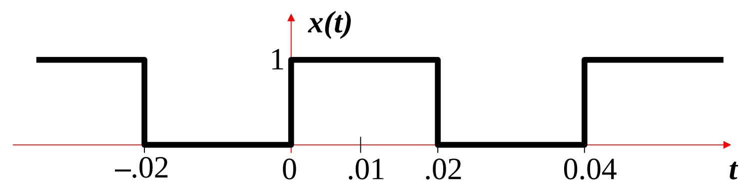
Integral is zero except for $k = \ell$

$$\Rightarrow a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt}\,dt$$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \tfrac{1}{2}T_0 \\ 0 & \tfrac{1}{2}T_0 \le t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



FS for a SQUARE WAVE $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k \neq 0)$$

$$a_k = \frac{1}{.04}\int_0^{.02} 1e^{-j(2\pi/.04)kt}dt = \frac{1}{.04(-j2\pi k/.04)}e^{-j(2\pi/.04)kt}\Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)}(e^{-j(\pi)k} - 1) = \frac{1-(-1)^k}{j2\pi k}$$

DC Coefficient: a_0

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k = 0)$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt = \frac{1}{T_0}(\text{Area})$$

$$a_0 = \frac{1}{.04}\int_0^{.02} 1dt = \frac{1}{.04}(.02 - 0) = \tfrac{1}{2}$$

Fourier Coefficients a_k

- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

$$a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

$$\omega_0 = 2\pi/(0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \frac{-j}{\pi k} & k = \pm1, \pm3, \ldots \\ 0 & k = \pm2, \pm4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$