

Lecture 14

Z Transforms: Introduction

7-Mar-05

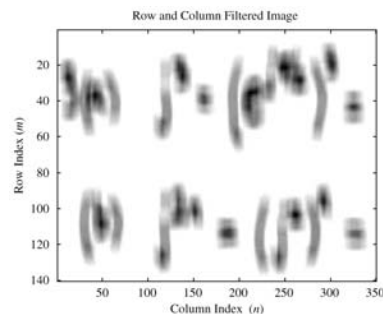
Quiz #2 Results

Graded out of: 100.0 Highest grade: 100.0 Mean grade: 80.4
 Number of records: 165 Lowest grade: 21.0 Median grade: 83.0

Score Range	Frequency
[55, 60)	4
[60, 65)	7
[65, 70)	15
[70, 75)	12
[75, 80)	18
[80, 85)	29
[85, 90)	30
[90, 95)	21
[95, 100)	22
[100]	1

FILTER TYPES

- LOW-PASS FILTER (**LPF**)
 - BLURRING
 - ATTENUATES HIGH FREQUENCIES
- HIGH-PASS FILTER (**HPF**)
 - SHARPENING for IMAGES
 - BOOSTS THE HIGHS
 - REMOVES DC
- BAND-PASS FILTER (**BPF**)



Superficial Knowledge

- It depends how carefully you think about it. If you don't think very carefully it's obvious; but if you think about it in depth, you'll get confused and it won't be obvious.

READING ASSIGNMENTS

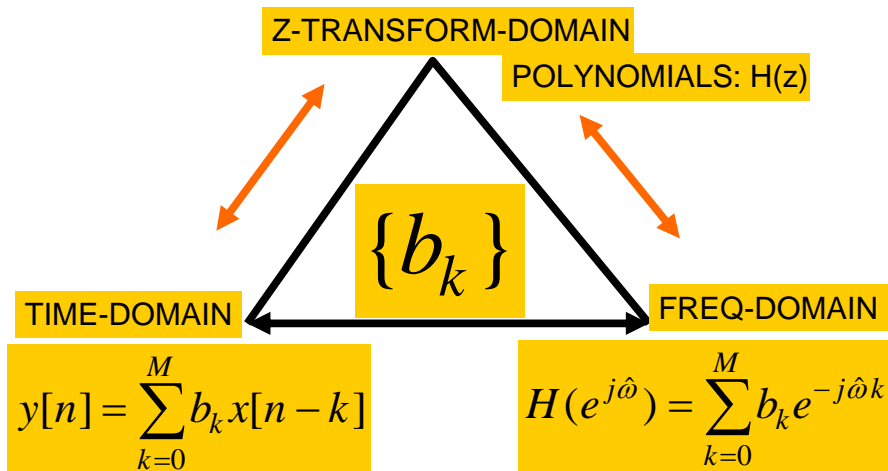
- This Lecture:
 - Chapter 7, Sects 7-1 through 7-5
- Other Reading:
 - Recitation: Ch. 7
 - CASCADING SYSTEMS
 - Next Lecture: Chapter 7, 7-6 to the end

LECTURE OBJECTIVES

- INTRODUCE the Z-TRANSFORM
 - Give Mathematical Definition
 - Show how the $H(z)$ POLYNOMIAL simplifies analysis
 - **CONVOLUTION** is SIMPLIFIED !
- Z-Transform can be applied to
 - FIR Filter: $h[n] \rightarrow H(z)$
 - Signals: $x[n] \rightarrow X(z)$

$$H(z) = \sum_n h[n]z^{-n}$$

TWO (no, THREE) DOMAINS

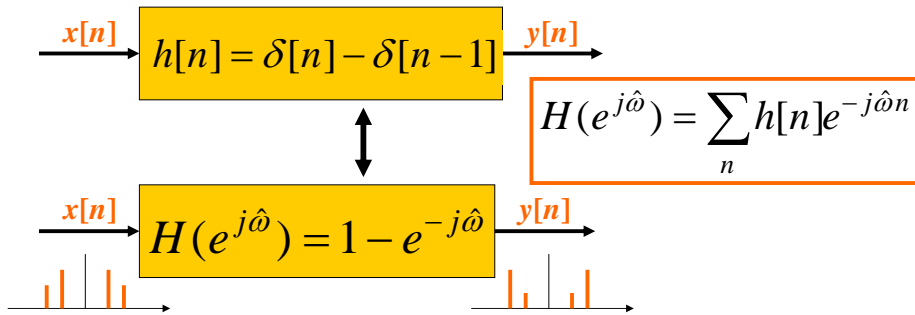


TRANSFORM CONCEPT

- Move to a new domain where
 - OPERATIONS are EASIER & FAMILIAR
 - Use POLYNOMIALS
- TRANSFORM both ways
 - $x[n] \rightarrow X(z)$ (into the z domain)
 - $X(z) \rightarrow x[n]$ (back to the time domain)

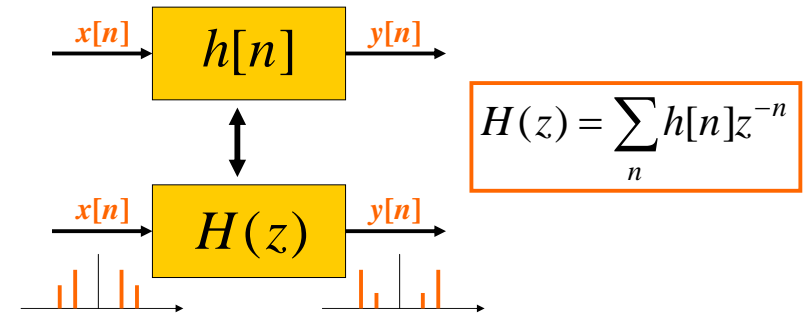
"TRANSFORM" EXAMPLE

- Equivalent Representations



Z-TRANSFORM IDEA

- POLYNOMIAL** REPRESENTATION



Z-Transform DEFINITION

- POLYNOMIAL Representation of LTI SYSTEM: $H(z) = \sum_n h[n]z^{-n}$

- EXAMPLE:

$$\{h[n]\} = \{2, 0, -3, 0, 2\}$$

$$H(z) = 2z^{-0} + 0z^{-1} - 3z^{-2} + 0z^{-3} + 2z^{-4}$$

$$= 2 - 3z^{-2} + 2z^{-4}$$

$$= 2 - 3(z^{-1})^2 + 2(z^{-1})^4$$

APPLIES to Any SIGNAL

POLYNOMIAL in z^{-1}

Z-Transform EXAMPLE

- ANY SIGNAL has a z-Transform:

$$X(z) = \sum_n x[n]z^{-n}$$

Example 7.1

n	$n < -1$	-1	0	1	2	3	4	5	$n > 5$
$x[n]$	0	0	2	4	6	4	2	0	0

$$X(z) = ? \quad X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4}$$

Example 7.2

$$X(z) = 1 - 2z^{-1} + 3z^{-3} - z^{-5}$$

EXPONENT GIVES TIME LOCATION

$$x[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ -2 & n = 1 \\ 0 & n = 2 \\ 3 & n = 3 \\ 0 & n = 4 \\ -1 & n = 5 \\ 0 & n > 5 \end{cases}$$

$$x[n] = ?$$

$$x[n] = \delta[n] - 2\delta[n-1] + 3\delta[n-3] - \delta[n-5]$$

Z-Transform of FIR Filter

- CALLLED the **SYSTEM FUNCTION**

- h[n] is same as {b_k}

SYSTEM FUNCTION

$$H(z) = \sum_{k=0}^M b_k z^{-k} = \sum_{k=0}^M h[k] z^{-k}$$

$$y[n] = \sum_{k=0}^M b_k x[n-k] = \sum_{k=0}^M h[k] x[n-k]$$

FIR DIFFERENCE EQUATION

CONVOLUTION

Z-Transform of FIR Filter

- Get H(z) DIRECTLY from the {b_k}
- Example 7.3 in the book:

$$y[n] = 6x[n] - 5x[n-1] + x[n-2]$$

$$\{b_k\} = \{6, -5, 1\}$$

$$H(z) = \sum b_k z^{-k} = 6 - 5z^{-1} + z^{-2}$$

Ex. DELAY SYSTEM

- UNIT DELAY: find h[n] and H(z)

$$x[n] \rightarrow \delta[n-1] \rightarrow y[n] = x[n-1]$$

$$H(z) = \sum \delta[n-1] z^{-n} = z^{-1}$$

$$x[n] \rightarrow z^{-1} \rightarrow y[n]$$

DELAY EXAMPLE

- UNIT DELAY: find $y[n]$ via polynomials
 - $x[n] = \{3, 1, 4, 1, 5, 9, 0, 0, 0, \dots\}$

$$Y(z) = z^{-1}X(z)$$

$$Y(z) = z^{-1}(3 + z^{-1} + 4z^{-2} + z^{-3} + 5z^{-4} + 9z^{-5})$$

$$Y(z) = 0z^0 + 3z^{-1} + z^{-2} + 4z^{-3} + z^{-4} + 5z^{-5} + 9z^{-6}$$

n	$n < 0$	0	1	2	3	4	5	6	$n > 6$
$y[n]$	0	0	3	1	4	1	5	9	0

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DELAY PROPERTY

A delay of one sample multiplies the z-transform by z^{-1} .

$$x[n - 1] \iff z^{-1}X(z)$$

Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n - n_0] \iff z^{-n_0}X(z)$$

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GENERAL I/O PROBLEM

- Input is $x[n]$, find $y[n]$ (for FIR, $h[n]$)
- How to combine $X(z)$ and $H(z)$?

Example 7.5

$$x[n] = \delta[n - 1] - \delta[n - 2] + \delta[n - 3] - \delta[n - 4]$$

$$\text{and } h[n] = \delta[n] + 2\delta[n - 1] + 3\delta[n - 2] + 4\delta[n - 3]$$

$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$

$$\text{and } H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

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FIR Filter = CONVOLUTION

$x[n], X(z)$	0	+1	-1	+1	-1					
$h[n], H(z)$	1	2	3	4						

	0	+1	-1	+1	-1					
		0	+2	-2	+2	-2				
			0	+3	-3	+3	-3			
					0	+4	-4	+4	-4	

$y[n], Y(z)$	0	+1	+1	+2	+2	-3	+1	-4		

$$y[n] = \sum_{k=0}^M b_k x[n - k] = \sum_{k=0}^M h[k] x[n - k]$$

CONVOLUTION

CONVOLUTION PROPERTY

■ PROOF:

$$y[n] = x[n] * h[n] = \sum_{k=0}^M h[k]x[n - k]$$

$$Y(z) = \sum_{k=0}^M h[k] (z^{-k} X(z))$$

MULTIPLY Z-TRANSFORMS

$$= \left(\sum_{k=0}^M h[k]z^{-k} \right) X(z) = H(z)X(z).$$

CONVOLUTION EXAMPLE

■ **MULTIPLY** the z-TRANSFORMS:

Example 7.5

$$x[n] = \delta[n - 1] - \delta[n - 2] + \delta[n - 3] - \delta[n - 4]$$

and $h[n] = \delta[n] + 2\delta[n - 1] + 3\delta[n - 2] + 4\delta[n - 3]$

$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$

and $H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$

MULTIPLY $H(z)X(z)$

CONVOLUTION EXAMPLE

- Finite-Length input $x[n]$
- FIR Filter ($L=4$)

MULTIPLY Z-TRANSFORMS

$$Y(z) = H(z)X(z)$$

$$= (1 + 2z^{-1} + 3z^{-2} + 4z^{-3})(z^{-1} - z^{-2} + z^{-3} - z^{-4})$$

$$= z^{-1} + (-1 + 2)z^{-2} + (1 - 2 + 3)z^{-3} + (-1 + 2 - 3 + 4)z^{-4}$$

$$+ (-2 + 3 - 4)z^{-5} + (-3 + 4)z^{-6} + (-4)z^{-7}$$

$$= z^{-1} + z^{-2} + 2z^{-3} + 2z^{-4} - 3z^{-5} + z^{-6} - 4z^{-7}$$

$y[n] = ?$

CASCADE SYSTEMS

- Does the order of S_1 & S_2 matter?
 - NO, **LTI SYSTEMS can be rearranged !!!**
 - Remember: $h_1[n] * h_2[n]$
 - How to combine $H_1(z)$ and $H_2(z)$?

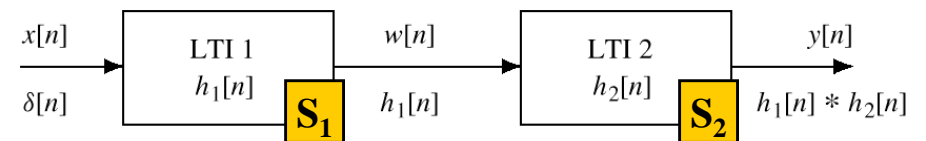
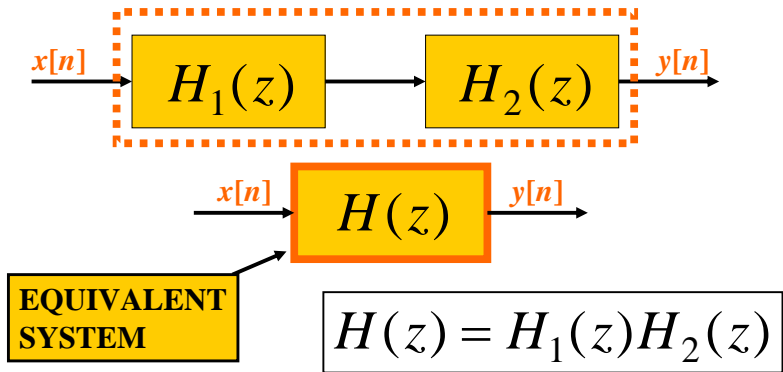


Figure 5.19 A Cascade of Two LTI Systems.

CASCADE EQUIVALENT

- Multiply the System Functions



CASCADE EXAMPLE

