

## Lecture 21

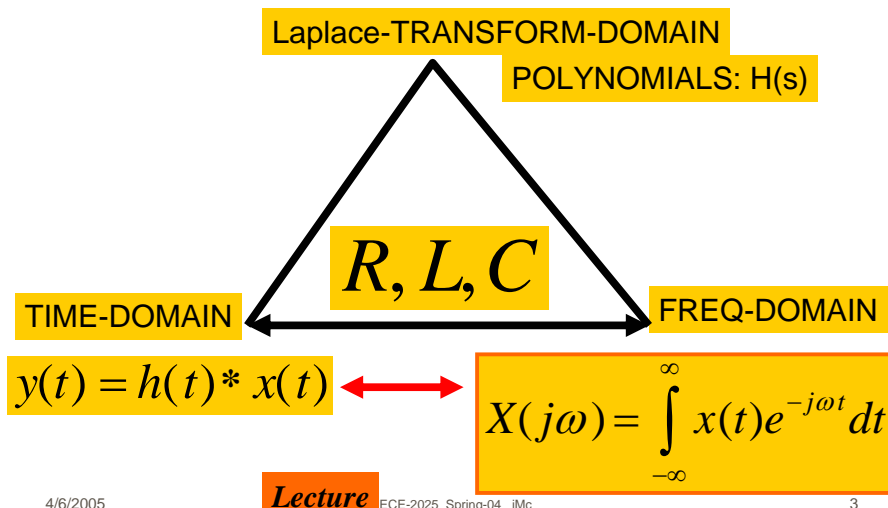
## Introduction to the Fourier Transform

11-April-05

## Info: Web-CT, Lab, HW

- **CHECK YOUR GRADES !!!**
  - Web-CT is the OFFICIAL gradebook
  - **Quiz #2 grading issues: today is the last day**
- **Quiz #4 will be 22-April (Friday)**
  - Coverage: HW #11, 12 and 13
  - Chapters 9, 10, and 11
  - Review Session, 21-April, Thurs @ 6:00 pm
- HW #12 due this week
- Lab #10 (Formal) will be due starting 13-April
  - Lab #10 requires a test/demo
  - TouchTone Decoder

## THREE DOMAINS: ANALOG



## READING ASSIGNMENTS

- This Lecture:
  - Chapter 11, Sects. 11-1 to 11-4
- Other Reading:
  - Recitation: Ch. 10
    - And Chapter 11, Sects. 11-1 to 11-4
  - Next Lecture: Chapter 11, Sects. 11-5, 11-6

# LECTURE OBJECTIVES

- Review
  - Frequency Response
  - Fourier Series
- Definition of **Fourier transform**

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt$$

Relation to Fourier Series

- Examples of Fourier transform pairs

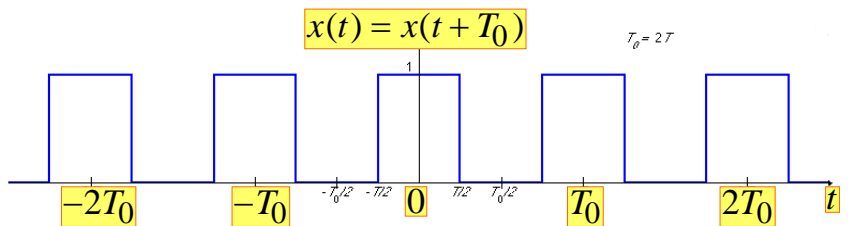
# Everything = Sum of Sinusoids

- One Square Pulse = Sum of Sinusoids
  - ????????????
- Finite Length
- Not Periodic



- Limit of Square Wave as Period  $\rightarrow$  infinity
  - Intuitive Argument

# Fourier Series: Periodic $x(t)$



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

**Fourier Synthesis**

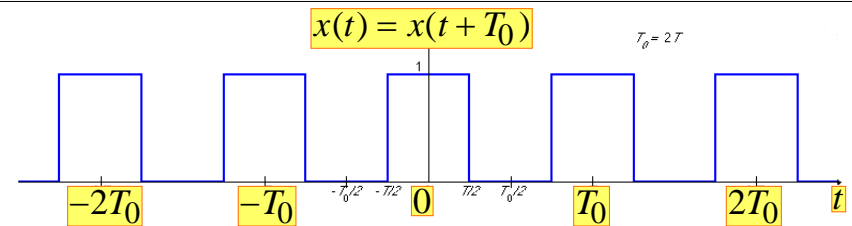
Fundamental Freq.

$$\omega_0 = 2\pi / T_0 = 2\pi f_0$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)e^{-j\omega_0 k t} dt$$

**Fourier Analysis**

# Square Wave Signal



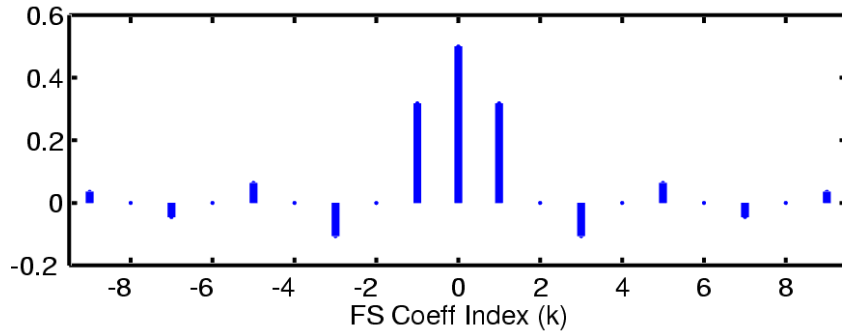
$$a_k = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} (1)e^{-j\omega_0 k t} dt$$

$$a_k = \frac{e^{-j\omega_0 k t} \Big|_{-T_0/4}^{T_0/4}}{-j\omega_0 k T_0} = \frac{e^{-j\pi k / 2} - e^{j\pi k / 2}}{-j2\pi k} = \frac{\sin(\pi k / 2)}{\pi k}$$

# Spectrum from Fourier Series

$$a_k = \frac{\sin(\pi k / 2)}{\pi k} = \begin{cases} \neq 0 & k = 0, \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \end{cases}$$

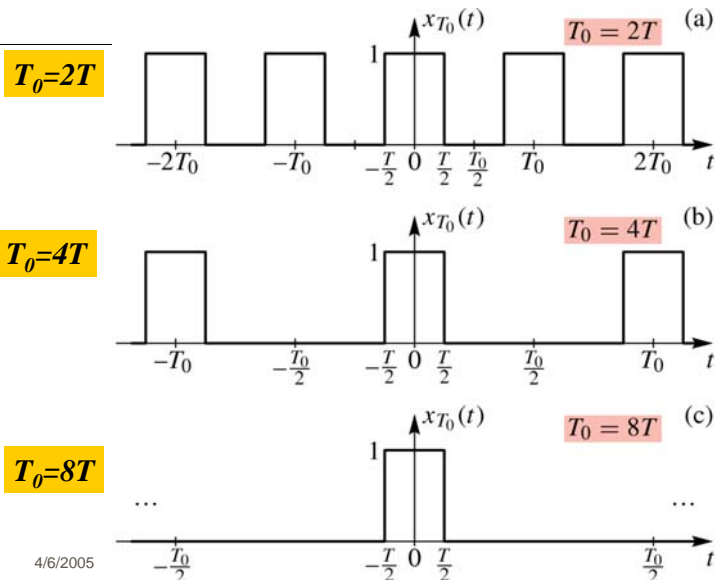
Fourier Series Coeffs for Square Wave



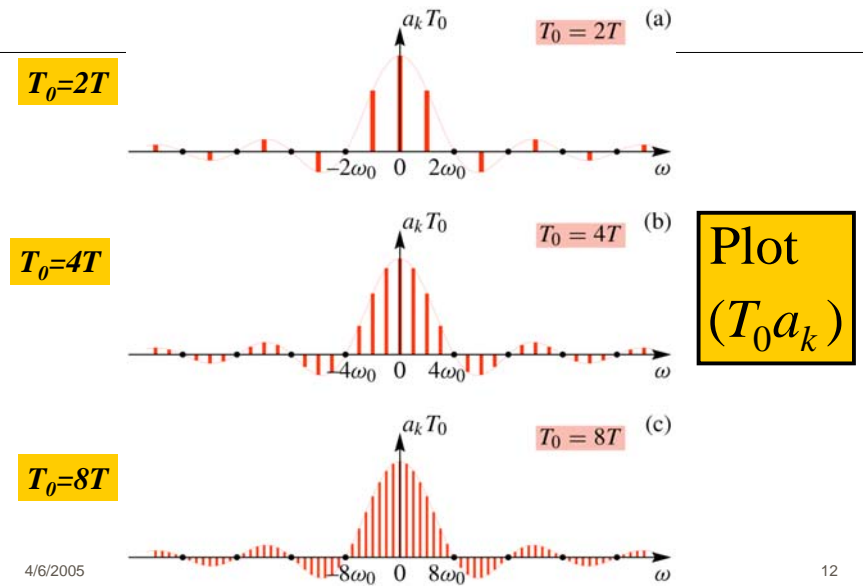
# What if $x(t)$ is not periodic?

- Sum of Sinusoids?
  - Non-harmonically related sinusoids
  - Would not be periodic, but would probably be non-zero for all  $t$ .
- Fourier transform
  - gives a “sum” (actually an **integral**) that involves **ALL** frequencies
  - can represent signals that are identically zero for negative  $t$ . !!!!!!!!

# Limiting Behavior of FS



# Limiting Behavior of Spectrum



## FS in the LIMIT (long period)

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} (T_0 a_k) e^{j\omega_0 k t} \left(\frac{2\pi}{T_0}\right) \mapsto x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

**Fourier Synthesis**

$$\lim_{T_0 \rightarrow \infty} \frac{2\pi}{T_0} = d\omega$$

$$\lim_{T_0 \rightarrow \infty} \frac{2\pi}{T_0} k = \omega$$

$$\lim_{T_0 \rightarrow \infty} T_0 a_k = X(j\omega)$$

$$T_0 a_k = \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_0 k t} dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

**Fourier Analysis**

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## Fourier Transform Defined

- For non-periodic signals

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

**Fourier Synthesis**

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

**Fourier Analysis**

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Example 1:

$$x(t) = e^{-at} u(t)$$

$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$X(j\omega) = -\frac{e^{-at} e^{-j\omega t}}{a+j\omega} \Big|_0^{\infty} = \frac{1}{a+j\omega} \quad a > 0$$

$$X(j\omega) = \frac{1}{a+j\omega}$$

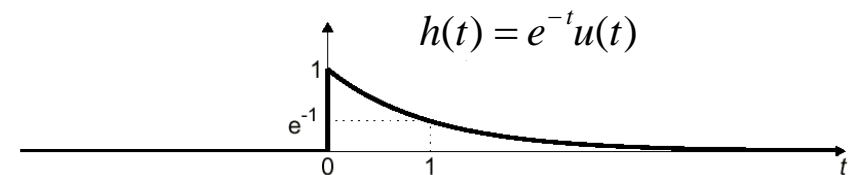
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## Frequency Response

- Fourier Transform of  $h(t)$  is the Frequency Response



$$h(t) = e^{-t} u(t) \Leftrightarrow H(j\omega) = \frac{1}{1+j\omega}$$

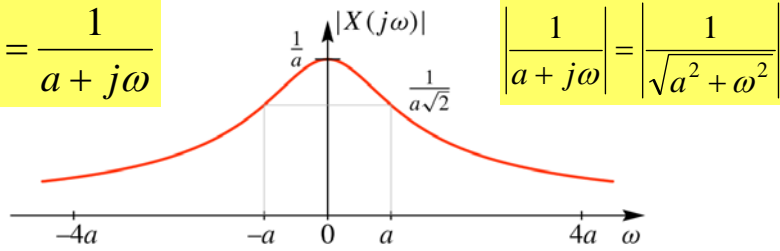
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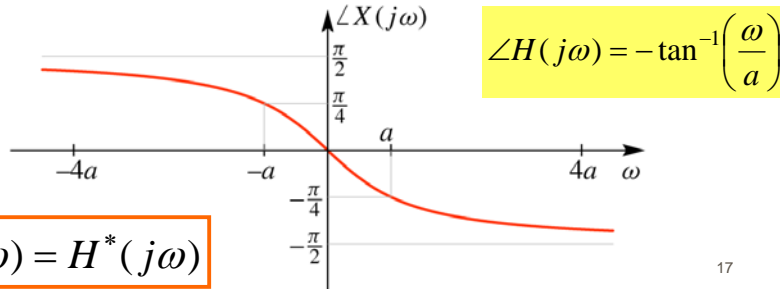
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# Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{a + j\omega}$$



$$\left| \frac{1}{a + j\omega} \right| = \frac{1}{\sqrt{a^2 + \omega^2}}$$



$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

$$H(-j\omega) = H^*(j\omega)$$

# Example 2:

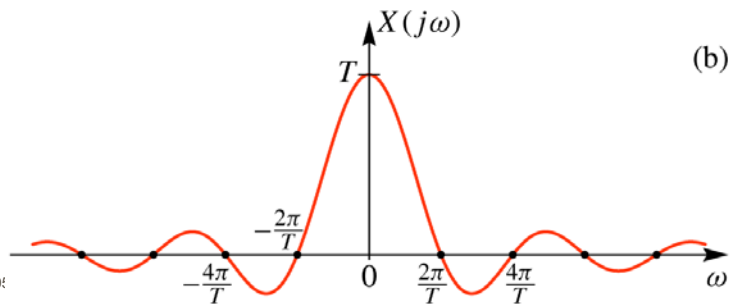
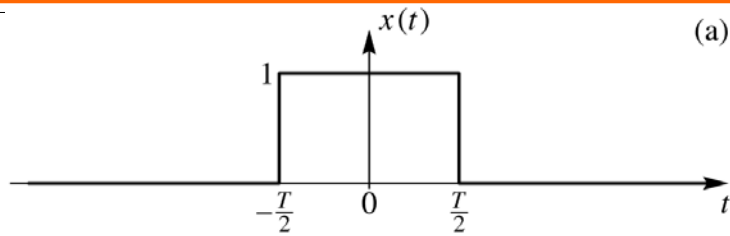
$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}$$

$$X(j\omega) = \int_{-T/2}^{T/2} (1)e^{-j\omega t} dt = \int_{-T/2}^{T/2} e^{-j\omega t} dt$$

$$X(j\omega) = \frac{e^{-j\omega t}}{-j\omega} \Big|_{-T/2}^{T/2} = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$X(j\omega) = \frac{\sin(\omega T / 2)}{(\omega / 2)}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T / 2)}{(\omega / 2)}$$



# Example 3:

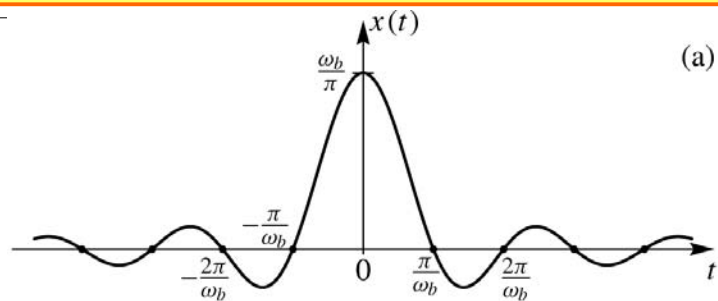
$$X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} e^{j\omega t} d\omega$$

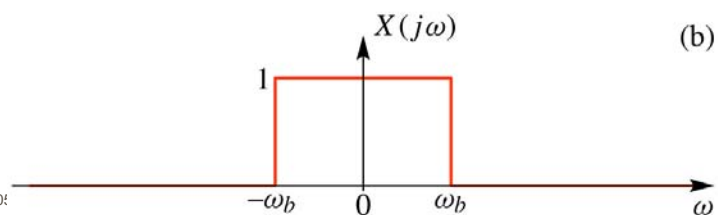
$$x(t) = \frac{1}{2\pi} \frac{e^{j\omega t}}{jt} \Big|_{-\omega_0}^{\omega_0} = \frac{1}{2\pi} \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{jt}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$



(a)



(b)

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Example 4:  $x(t) = \delta(t - t_0)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

**Shifting Property of the Impulse**

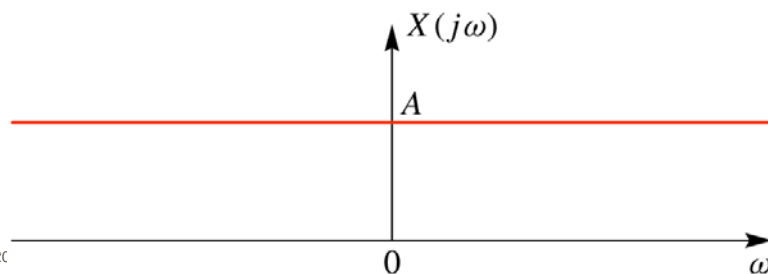
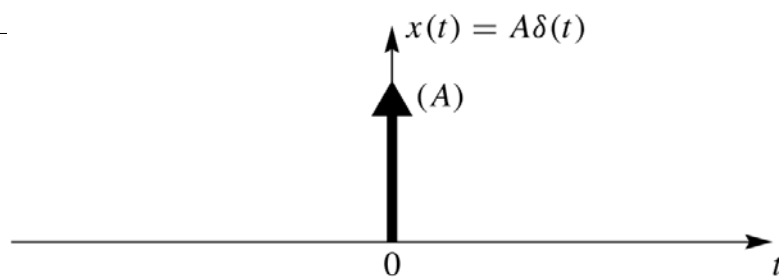
$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0}$$

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$$x(t) = \delta(t) \Leftrightarrow X(j\omega) = 1$$



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0

omega

Example 5:  $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0) e^{j\omega t} d\omega = e^{j\omega_0 t}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)$$

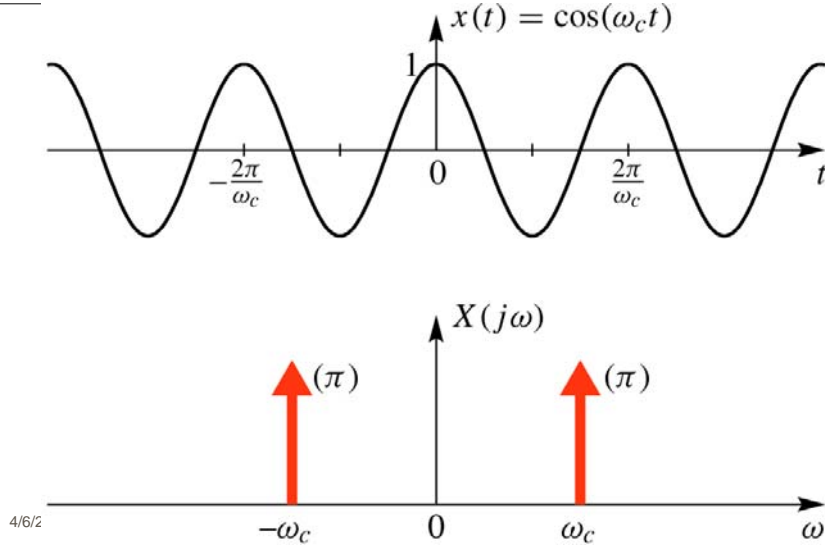
$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$

$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

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$$x(t) = \cos(\omega_0 t) \Leftrightarrow X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



## Table of Fourier Transforms

$$x(t) = e^{-at} u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T / 2)}{(\omega / 2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$