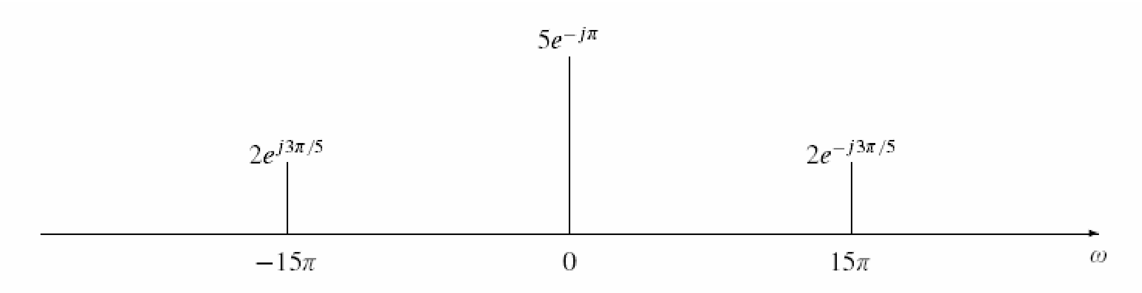
SOLUTIONS Homework 3 ECE2025 Spring 2005

Problem 3.1
The spectrum for *x(t)* is given as



(a) Write an equation for x(t) as a sum of cosines.

The middle spectral line corresponds to the "DC component" with value -5. The other two components correspond to the sum
$2\left(e^{j(15\pi t-3\pi/5)}+e^{-j(15\pi t-3\pi/5)}\right)$.
This latter expression can be put into the form of the inverse Euler formula (2.17) by multiplying and dividing by 2:
$$4\left(\frac{e^{j(15\pi t-3\pi/5)}+e^{-j(15\pi t-3\pi/5)}}{2}\right)=4\cos(15\pi t-3\pi/5)$$
Therefore, the complete expression for x(t) is:
$$x(t)=-5+4\cos(15\pi t-3\pi/5)$$

(b) Plot the spectrum of the signal y(t) defined as: $y(t)=x^2(t)-7$.

$$\begin{aligned}y(t)&=x^2(t)-7\\&=(-5+4\cos(15\pi t-3\pi/5))^2-7\\&=25-7-40\cos(15\pi t-3\pi/5)+16\cos^2(15\pi t-3\pi/5)\end{aligned}$$

Using the trigonometric identity $\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$, we can simplify y(t) to
$$\begin{aligned}y(t)&=18-40\cos(15\pi t-3\pi/5)+8(1+\cos(30\pi t-6\pi/5))\\&=26-40\cos(15\pi t-3\pi/5)+8\cos(30\pi t-6\pi/5)\end{aligned}$$
Substituting the Inverse Euler Formula for cos yields

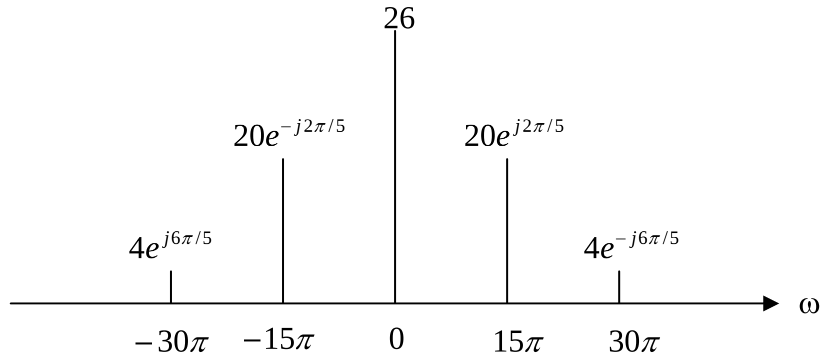
$$\begin{aligned}y(t)&=26-40\left(\frac{e^{j(15\pi t-3\pi/5)}+e^{-j(15\pi t-3\pi/5)}}{2}\right)+8\left(\frac{e^{j(30\pi t-6\pi/5)}+e^{-j(30\pi t-6\pi/5)}}{2}\right)\\&=26-20\left(e^{j(15\pi t-3\pi/5)}+e^{-j(15\pi t-3\pi/5)}\right)+4\left(e^{j(30\pi t-6\pi/5)}+e^{-j(30\pi t-6\pi/5)}\right)\\&=26-20e^{j(15\pi t-3\pi/5)}-20e^{-j(15\pi t-3\pi/5)}+4e^{j(30\pi t-6\pi/5)}+4e^{-j(30\pi t-6\pi/5)}\end{aligned}$$

Finally, the negative sign on the coefficients should be represented as a phase shift:

$$y(t) = 26 + 20e^{j\pi}e^{j(15\pi t - 3\pi/5)} + 20e^{-j\pi}e^{-j(15\pi t - 3\pi/5)} + 4e^{j(30\pi t - 6\pi/5)} + 4e^{-j(30\pi t - 6\pi/5)}$$

$$= 26 + 20e^{j(15\pi t + 2\pi/5)} + 20e^{-j(15\pi t + 2\pi/5)} + 4e^{j(30\pi t - 6\pi/5)} + 4e^{-j(30\pi t - 6\pi/5)}$$

There will be a spectral line for each term:



(Alternatively, this expression could have been obtained by substituting the exponential version of x(t): $y(t) = \left[-5 + 2\left(e^{j(15\pi t - 3\pi/5)} + e^{-j(15\pi t - 3\pi/5)}\right)\right]^2 - 7$.)

(c) Plot the spectrum of the real-valued signal $z(t) = 10x(t)\cos(7.5\pi t + 2\pi/5)$.

$$z(t) = 10[-5 + 4\cos(15\pi t - 3\pi/5)]\cos(7.5\pi t + 2\pi/5)$$

$$= -50\cos(7.5\pi t + 2\pi/5) + 40\cos(15\pi t - 3\pi/5)\cos(7.5\pi t + 2\pi/5)$$

Using the trig identity cosAcosB=1/2cos(A+B) + 1/2cos(A-B) on the product of cosines term yields

$$40\cos(15\pi t - 3\pi/5)\cos(7.5\pi t + 2\pi/5)$$

$$= 20\cos(15\pi t - 3\pi/5 + 7.5\pi t + 2\pi/5) + 20\cos(15\pi t - 3\pi/5 - 7.5\pi t - 2\pi/5)$$

$$= 20\cos(22.5\pi t - \pi/5) + 20\cos(7.5\pi t - \pi)$$

Therefore,

$$z(t) = -50\cos(7.5\pi t + 2\pi/5) + 20\cos(22.5\pi t - \pi/5) + 20\cos(7.5\pi t - \pi).$$

Using Inverse Euler yields

$$z(t) = -50\left(\frac{e^{j(7.5\pi t + 2\pi/5)} + e^{-j(7.5\pi t + 2\pi/5)}}{2}\right) + 20\left(\frac{e^{j(22.5\pi t - \pi/5)} + e^{-j(22.5\pi t - \pi/5)}}{2}\right)$$

$$+ 20\left(\frac{e^{j(7.5\pi t - \pi)} + e^{-j(7.5\pi t - \pi)}}{2}\right)$$

Simplifying and letting -1=exp($j\pi$) gives

$$z(t) = 25\left(e^{j(7.5\pi t + 7\pi/5)} + e^{-j(7.5\pi t + 7\pi/5)}\right) + 10\left(e^{j(22.5\pi t - \pi/5)} + e^{-j(22.5\pi t - \pi/5)}\right)$$

$$+ 10\left(e^{j(7.5\pi t - \pi)} + e^{-j(7.5\pi t - \pi)}\right)$$

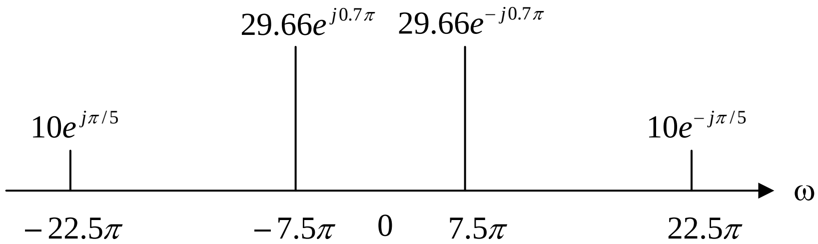
We observe we have some rotating phasors of the same frequency; these can be combined.

$$25e^{j(7.5\pi t+7\pi/5)}+10e^{j(7.5\pi t-\pi)}=\left[25e^{j7\pi/5}-10\right]e^{j7.5\pi t}$$

$$=29.6565\mathrm{e}^{-j0.7039\pi}$$

Then,

$$z(t)=29.6565\left(e^{j(7.5\pi t-0.7039\pi)}+e^{-j(7.5\pi t-0.7039\pi)}\right)+10\left(e^{j(22.5\pi t-\pi/5)}+e^{-j(22.5\pi t-\pi/5)}\right)$$



Problem 3.2

Given:

$x_i(t) = x_s(t) + x_d(t)\cos(\omega_c t + \psi)$,

where

$x_s(t) = x_R(t) + x_L(t)$

$x_d(t) = x_R(t) - x_L(t)$.

(a) Suppose $x_L(t) = 7\cos(10000\pi t - \pi/3)$, $x_R(t) = 0$, and $\psi = 0$. Draw the spectrum of $x_i(t)$.

Making these substitutions, we have

$$x_i(t) = 7\cos(10000\pi t - \pi/3) - 7\cos(10000\pi t - \pi/3)\cos(\omega_c t)$$

Substituting the inverse Euler formulas,

$$x_i(t) = 7\left(\frac{e^{j(10000\pi t - \pi/3)} + e^{-j(10000\pi t - \pi/3)}}{2}\right)$$
$$-7\left(\frac{e^{j(10000\pi t - \pi/3)} + e^{-j(10000\pi t - \pi/3)}}{2}\right)\left(\frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}\right)$$
$$= \frac{7}{2}\left(e^{j(10000\pi t - \pi/3)} + e^{-j(10000\pi t - \pi/3)}\right)$$
$$-\frac{7}{4}\left(e^{j([\omega_c + 10000\pi]t - \pi/3)} + e^{j([\omega_c - 10000\pi]t + \pi/3)} + e^{j([-\omega_c + 10000\pi]t - \pi/3)} + e^{j([-\omega_c - 10000\pi]t + \pi/3)}\right)$$

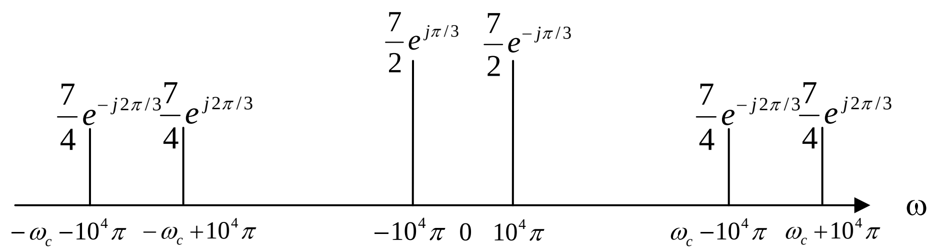
Next, take care of the negative sign:

$$x_i(t) = \frac{7}{2}\left(e^{j(10000\pi t - \pi/3)} + e^{-j(10000\pi t - \pi/3)}\right)$$
$$+\frac{7}{4}\left(e^{j([\omega_c + 10000\pi]t + 2\pi/3)} + e^{j([\omega_c - 10000\pi]t + 4\pi/3)} + e^{j([-\omega_c + 10000\pi]t + 2\pi/3)} + e^{j([-\omega_c - 10000\pi]t + 4\pi/3)}\right)$$

Use that 4π/3=-2 π /3:

$$x_i(t) = \frac{7}{2}\left(e^{j(10000\pi t - \pi/3)} + e^{-j(10000\pi t - \pi/3)}\right)$$
$$+\frac{7}{4}\left(e^{j([\omega_c + 10000\pi]t + 2\pi/3)} + e^{j([\omega_c - 10000\pi]t - 2\pi/3)} + e^{j([-\omega_c + 10000\pi]t + 2\pi/3)} + e^{j([-\omega_c - 10000\pi]t - 2\pi/3)}\right)$$

We observe that we have spectral lines at $\pm 10000\pi$, $\pm[\omega_c + 10000\pi]$, and $\pm[\omega_c - 10000\pi]$. Assuming $\omega_c >> 10000\pi$, the spectrum is



(b) Now let $x_L(t) = 7\cos(10000\pi t - \pi/3)$ and $x_R(t) = 6\cos(10000\pi t + 2\pi/3)$.
Find the spectrum of $x_i(t)$, assuming $\psi = 0$.

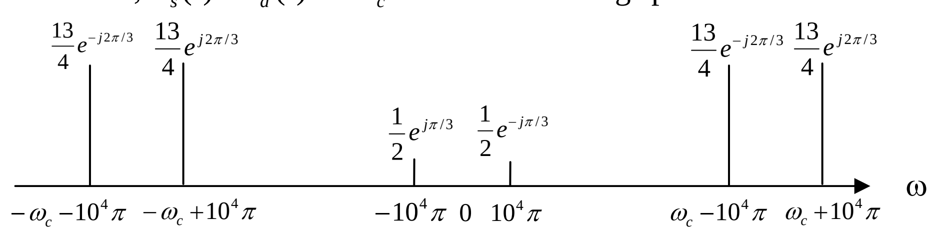
It will be convenient to first simplify $x_s(t)$ and $x_d(t)$:

$$x_s(t) = 7\cos(10000\pi t - \pi/3) + 6\cos(10000\pi t + 2\pi/3)$$
$$= \mathrm{Re}\left\{\left(7e^{-j\pi/3} + 6e^{j2\pi/3}\right)e^{j10000\pi t}\right\} = \mathrm{Re}\left\{e^{-j\pi/3}e^{j10000\pi t}\right\}$$
$$= \frac{e^{-j\pi/3}e^{j10000\pi t} + e^{j\pi/3}e^{-j10000\pi t}}{2}$$
$$x_d(t) = 6\cos(10000\pi t + 2\pi/3) - 7\cos(10000\pi t - \pi/3)$$
$$= \mathrm{Re}\left\{\left(6e^{j2\pi/3} - 7e^{-j\pi/3}\right)e^{j10000\pi t}\right\} = \mathrm{Re}\left\{13e^{j2\pi/3}e^{j10000\pi t}\right\}$$

Next, multiply $x_d(t)$ by the cosine:

$$x_d(t)\cos\omega_c t = 13\left(\frac{e^{j2\pi/3}e^{j10000\pi t} + e^{-j2\pi/3}e^{-j10000\pi t}}{2}\right)\left(\frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}\right)$$
$$= \frac{13}{4}\left(e^{j2\pi/3}e^{j(10000\pi+\omega_c)t} + e^{-j2\pi/3}e^{j(-10000\pi+\omega_c)t} + e^{j2\pi/3}e^{j(10000\pi-\omega_c)t} + e^{-j2\pi/3}e^{j(-10000\pi-\omega_c)t}\right)$$

Therefore, $x_s(t) + x_d(t)\cos\omega_c t$ has the following spectrum.



Problem 3.3.

(a) The spectra of real signals have conjugate symmetry. That is, if a spectral line at ω_o radians/sec has a complex amplitude of $\alpha e^{j\beta}$, then there will be a spectral line at $-\omega_o$ with complex amplitude $\alpha e^{-j\beta}$. The table specifies five spectral lines. Therefore there will be two conjugate pairs of spectral lines. Since $\omega_3 > \omega_1$, then the lines at $-\omega_3$ and $3\omega_1$ must be a pair, and the lines at ω_1 and -100π must be another pair. That implies

$\omega_1 = 100\pi$

$\omega_3 = 300\pi$

The amplitudes of the lines at $-\omega_3$ and $3\omega_1$ must be conjugates of each other, so $3X_1e^{j\pi/3} = X_3^*$. Similarly, the amplitudes of the lines at ω_1 and -100π must be conjugates of each other, so $X_1 = -2 - j2 = X_{-1}^*$. Therefore, we have

$X_1 = -2 - j2 = \sqrt{8}e^{-j3\pi/4}$

$X_{-1} = -2 + j2 = \sqrt{8}e^{j3\pi/4}$

$X_3 = 3X_1^*e^{-j\pi/3} = 3(-2 + j2)e^{-j\pi/3} = 3\left(\sqrt{8}e^{j3\pi/4}\right)e^{-j\pi/3} = 3\sqrt{8}e^{j5\pi/12}$

$X_{-3} = 3\sqrt{8}e^{-j5\pi/12}$

(b) To reconstruct x(t), we add the following terms:

$$\begin{aligned} x(t) &= B + X_1e^{j\omega_1 t} + X_{-1}e^{-j\omega_1 t} + X_3e^{j\omega_3 t} + X_{-3}e^{-j\omega_3 t} \\ &= B + \left(\sqrt{8}e^{-j3\pi/4}e^{j\omega_1 t} + \sqrt{8}e^{j3\pi/4}e^{-j\omega_1 t}\right) \\ &\quad + \left(3\sqrt{8}e^{j5\pi/12}e^{j\omega_3 t} + 3\sqrt{8}e^{-j5\pi/12}e^{-j\omega_3 t}\right) \\ &= -3 + 2\sqrt{8}\cos(\omega_1 t) + 6\sqrt{8}\cos(\omega_3 t) \\ &= -3 + 2\sqrt{8}\cos(100\pi t) + 6\sqrt{8}\cos(300\pi t) \end{aligned}$$

(c) The fundamental period is the reciprocal of the fundamental frequency. The fundamental frequency is the greatest common divisor of all the frequency components (i.e. the cosines) in the signal. $x(t)$ has two frequency components, 100π rad/s and 300π rad/s, or 50 Hz and 150 Hz, respectively. The greatest common divisor is 50 Hz. Therefore, the fundamental period is 1/50 s or 0.02 s.

Problem 3.4

$$
\begin{aligned}
x(t) &= 0.8A_2\cos(2\pi 436t + 2\pi/3) + A_2\cos(2\pi 440t + 2\pi/3) + 0.8A_2\cos(2\pi 444t + 2\pi/3) \\
&= A_2\left[\mathrm{Re}\left\{0.8e^{j(2\pi 436t + 2\pi/3)}\right\} + \mathrm{Re}\left\{e^{j(2\pi 440t + 2\pi/3)}\right\} + \mathrm{Re}\left\{0.8e^{j(2\pi 444t + 2\pi/3)}\right\}\right] \\
&= A_2\left[0.8\,\mathrm{Re}\left\{e^{j(2\pi 436t + 2\pi/3)} + e^{j(2\pi 444t + 2\pi/3)}\right\} + \mathrm{Re}\left\{e^{j(2\pi 440t + 2\pi/3)}\right\}\right] \\
&= A_2\left[0.8\,\mathrm{Re}\left\{e^{j(2\pi[440-4]t + 2\pi/3)} + e^{j(2\pi[440+4]t + 2\pi/3)}\right\} + \mathrm{Re}\left\{e^{j(2\pi 440t + 2\pi/3)}\right\}\right] \\
&= A_2\left[0.8\,\mathrm{Re}\left\{e^{j(2\pi 440t + 2\pi/3)}\left(e^{-j2\pi 4t} + e^{j2\pi 4t}\right)\right\} + \mathrm{Re}\left\{e^{j(2\pi 440t + 2\pi/3)}\right\}\right] \\
&= A_2\,\mathrm{Re}\left\{e^{j(2\pi 440t + 2\pi/3)}\left[0.8\left(e^{-j2\pi 4t} + e^{j2\pi 4t}\right) + 1\right]\right\} \\
&= A_2\,\mathrm{Re}\left\{e^{j(2\pi 440t + 2\pi/3)}\left[1.6\cos(2\pi 4t) + 1\right]\right\} \\
&= A_2\left[1.6\cos(2\pi 4t) + 1\right]\mathrm{Re}\left\{e^{j(2\pi 440t + 2\pi/3)}\right\} \\
&= A_2\left[1.6\cos(2\pi 4t) + 1\right]\cos(2\pi 440t + 2\pi/3)
\end{aligned}
$$

Therefore,

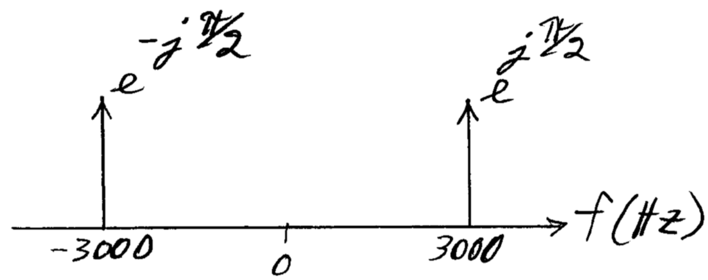
$e(t) = A_2\left[1.6\cos(2\pi 4t) + 1\right]$.

This envelope reaches its peak value of $2.6A_2$ every 0.25 seconds.

Problem 3.5:

$$x(t) = [v(t) + A]\cos 2\pi(750\cdot 10^3)t$$

(a) $$v(t) = 2\cos\left(2\pi(3000)t + \frac{\pi}{2}\right)$$



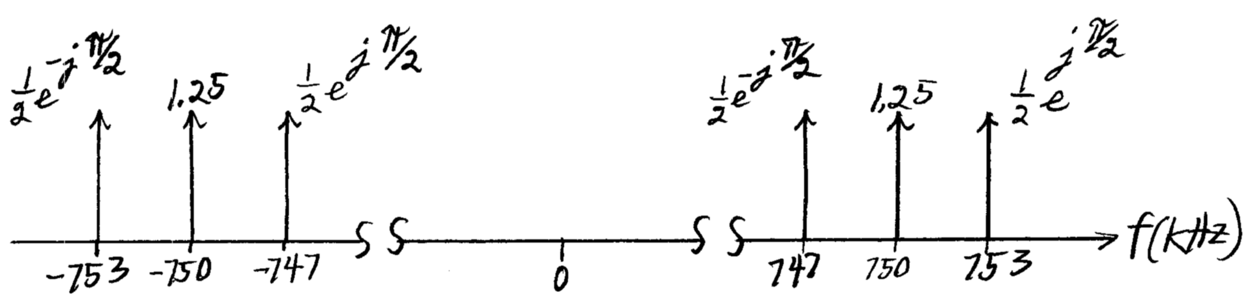
(b) $$x(t) = \left[2.5 + e^{j\left[2\pi(3000)t + \frac{\pi}{2}\right]} + e^{-j\left[2\pi(3000)t + \pi/2\right]}\right] \times$$

$$\frac{1}{2}\left[e^{j2\pi(750\cdot 10^3)t} + e^{-j2\pi(750\cdot 10^3)t}\right]$$

$$x(t) = 1.25e^{j2\pi(750\cdot 10^3)t} + 1.25e^{-j2\pi(750\cdot 10^3)t}$$

$$+ \frac{1}{2}e^{j[2\pi(753\cdot 10^3)t + \pi/2]} + \frac{1}{2}e^{-j[2\pi(753\cdot 10^3)t + \pi/2]}$$

$$+ \frac{1}{2}e^{j[2\pi(747\cdot 10^3)t - \pi/2]} + \frac{1}{2}e^{-j[2\pi(747\cdot 10^3)t - \frac{\pi}{2}]}$$



Problem 3.6*:

(a) #5. Period of (a) is 1s; fundamental frequency of #5 is $f_0 = 1$ Hz.

(b) #2. Period of (b) is 1.5s; frequency of #2 is $\omega = 3\pi$ rad/s. Phase of $\phi = -\pi$ means that $x(t)$ is at a negative peak at $t = 0$.

(c) #4. Period of (c) is 2s; fundamental frequency of #4 is $f_0 = 0.5$ Hz.

(d) #1. Period of (d) is 1s, DC value is +2, and $\phi > 0$; frequency of #1 is $f = 1$ Hz with positive phase.

(e) #3. Period of (d) is 1s, DC value is +2, and $\phi < 0$; frequency of #3 is $f = 1$ Hz with negative phase.

1. $x(t) = 2 + 3\cos(2\pi t + \pi/2)$
2. $x(t) = 3\cos(2\pi(1.5)t + \pi)$
3. $x(t) = 2 + 3\cos(2\pi t - 0.4\pi)$
4. $x(t) = 3\cos(\pi t + 0.8\pi) + 3\cos(2\pi(1.5)t + \pi)$
5. $x(t) = 3\cos(2\pi t - 0.4\pi) + 3\cos(4\pi t + \pi)$